# Voronoi Translated

## Introduction to Voronoi tessellation
## and essays by
## G L Dirichlet and G F Voronoi

Introduction and translation by

Kit Tiyapan

Kittisak Nui Tiyapan

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To Prof. Graham Davies
on his retirement

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# Preface

The three papers by Voronoi all appeared in the same journal, the influential and prestigious *Journal für die reine und angewandte Mathematik* [Journal for the pure and applied mathematics]. This journal was a leading mathematical journal during $19^{th}$ and early $20^{th}$ centuries when most of its publications appeared either in French or in German. In French it is called *Journal de Crelle* or Crelle's Journal after the name of its founder in 1826 by August Leopold Crelle (1780–1855). Originally Crelle intended the journal to emphasise equally both pure and applied mathematics. But the policy soon changed and it has been dealing solely with pure mathematics since the start of his second and short-lived *Journal für die Bankunst* (1829–1851) to deal the application side.

The main idea is quite simple. It is that space can be partitioned into a set of regions, each surrounding a single point which is sometimes called a nucleus. Every point in a given region is closer to its own nucleus than to any nuclei of other regions. This idea has found so many applications in nature that I think it is as beautiful as the golden ratio $\frac{1+\sqrt{5}}{2}$ is. It has been used to study the forest fire and it has been used to study the structure of the distribution of galaxy. In fact the internal structure of many things has proved to be Voronoi, things in nature as well as man-made ones, for example plant cells and filtering membranes. Looking at a two-dimensional Voronoi structure will remind one of plant or animal cells partitioned by straight walls which result from cosiness of circular cells growing among their neighbours.

All the pictures within this book and those on the cover have been either drawn or taken by myself. Those which have been drawn were produced on either Matlab, Mathcad, or Xfig.

I would like to thank the librarians and archivists of the John Ryland library at the University of Manchester where the original copy of the material has been heavily perused during the course of the translation. I would like to thank Emeritus Professor David J. Bell and Professor Graham A. Davies, both of UMIST, for introducing me to this interesting area of study. Also, I would like to thank Professor Davies further for his support and encouragement, without which this book would not have been possible.

Manchester, August 2001.

Georges Fedosevich Voronoi[†] was born in Zhuravka, Poltava guberniya, Russia (now Ukraine) on 28th April, 1868. His father was a superintendent of Gymnasiums (senior secondary schools) in Kishinëv, a town in Moldova now, and in other towns in the then southern Ukraine. He entered the Gymnasium in Priluki in 1885. After that he entered the Faculty of Physics and Mathematics at the University of St. Petersburg. He graduated from the department in 1885. Then he went on to do a master degree and completed it in 1894 His dissertation for the master degree was on algebraic integers associated with the roots of an irreducible cubic equation, that is a third-degree equation with a general form $ax^3 + bx^2 + cx + d = 0$. He became a professor in the Department of Pure Mathematics at University of Warsaw. His doctoral dissertation submitted to the University of St. Petersburg in 1897 was on a generalisation of the algorithm of continued fractions, that is fractions written in the form $r = \frac{1}{a+\frac{1}{b+\frac{1}{c+\ldots}}}$ and which terminates when $r$ is rational. In the work for his Ph.D. he provided an algorithm ofr calculating fundamental units of a general cubic field for both a positive and negative discriminant. Both of his dissertations won the Bunyakovsky prize[‡] by the St. Petersburg Academy of Science.

Voronoi's lifelong work was in theory of numbers and is divided into three groups, namely algebraic theory of numbers, analytic theory of numbers, and geometry of numbers. His three papers translated here make up two of the planned series of memoirs to apply the principal of Continuous Hermite (Charles Hermite, 1822–1901) parameters to problems of the arithmetical theory of definite and indefinite quadratic forms. He had completed only two of the series when he died in Warsaw, Poland on 20th November, 1908, before the last one appeared in print in 1909. A short obituary written by Kurt Hensel (1861–1941) was included at the end of the paper, which is also included here in translation.

The first paper gives characteristics of complete quadratic forms. In it Voronoi solved the question posed by Hermite on the upper limit of the minima of the positive quadratic forms for a given discriminant of $n$ variables. Zolotarev (Egor Ivanovich Zolotarev, 1847–1878) and Korkin (Aleksandr Nikolaevich Korkin) had given solutions for $n = 4$ and $n = 5$. Voronoi gave an algorithmic solution for any $n$. He did this with the help of the methods of the geometrical theory of numbers.

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[†] Sometimes written as Georgy Fedoseevich Voronoy. In French publications his last name is written *Voronoï*.

[‡] Named after Viktor Yakovlevich Bunyakovskii (1804–1889).

The second and the third papers deal with simple parallelepipeds, that is polyhedra with parallelograms as all their faces. He gave the determination of all possible methods of filling an $n$-dimensional Euclidean space with identical convex nonintersecting polyhedra (parallelepipeds) which have completely contiguous boundaries. A solution of this problem for three-dimensional space had been given by Fedorov (Evgraf Stepanovich Fyodorov, 1853–1919) who was a crystallographer although the proof he gave is said to be incomplete. Minkowski (Hermann Minkowski, 1864–1909) showed in 1896 that the parallelepipeds must have centres of symmetry. He also demonstrated that the number of their boundaries did not exceed $2(2^n - 1)$. Voronoi imposed further the requirement that $n + 1$ parallelepipeds converge at each summit and solved the problem for these conditions completely.

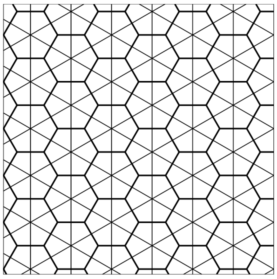
Voronoi's collected works appeared in three volumes under the title of *Sobranie sochineny*, Kiev, 1952–1953. He is sometimes referred to as belonging to the St. Petersburg school of the Theory of Numbers. This must have been the Petersburg Mathematical School sometimes called Chebyshev School or Petersburg School. It was founded by Chebyshev (Pafnuty Lvovich Chebyshev, 1821–1894) and had prominent figures as Grave (Dmitri Aleksandrovich Grave), Krylov (Aleksei Nikolaevich Krylov, 1863–1945), Lyapunov (Aleksandr Mikhailovich Lyapunov, 1857–1918), Markov (Andrei Andreyevich Markov, 1856–1922), Sohotski (Yulian-Karl Vasilievich Sokhotsky, 1842–1927), Steklov (Vladimir Andreevich Steklov, 1864–1926), Korkin, K. A. Posse, and A. V. Vassiliev.

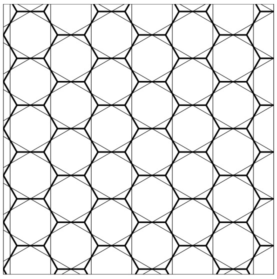
The topic of these three papers of Voronoi goes back to Gauss (Johann Carl Friedrich Gauss, 1777–1855) and Hermite

Tessellation or tiling is an arrangement of domains which fills the space. In two dimensions these domains are polygons and in three dimensions they are polyhedra. There are only three regular polygons that can tile. These are the equilateral triangle, the square and the regular hexagon, and they make up what is called *regular* tessellations. There are eight *semiregular* tessellations. The three regular ones comprise two pairs of dual lattices. The square is dual to itself. The regular hexagon and the equilateral triangle are dual to each other.

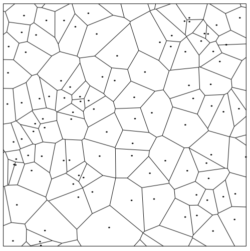
Apart from the dual lattice, one has a covering lattice which is defined to have vertices at the mid-points of edges in the original lattice and to have an edge wherever there used to be a vertex joining two edges. In the following two pictures the one on the left-hand side is hexagonal lattice and triangular lattice, which are dual to each other.

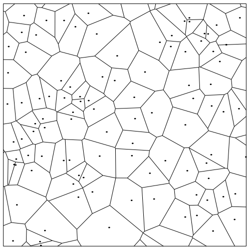
The one on the right-hand side is hexagonal lattice and its covering lattice called *Kagome* which comes from Japanese and which means a basket pattern. The word is pronounced differently from usual English pronunciation and is therefore written with an adopted *accent aigu* (acute accent) from French as *Kagomé*, perhaps because it is easier to do so than to explain to the reader. This is how the word literally became frenchised.

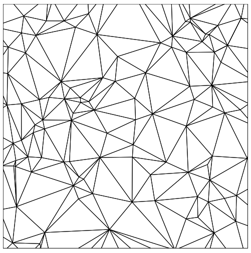

Dual lattices, hexagonal and triangular


Hexagonal and its covering lattice, Kagome

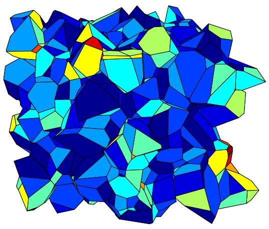
The following pictures show the Voronoi and its dual the Delaunay tessellations, as well as their superimposition.

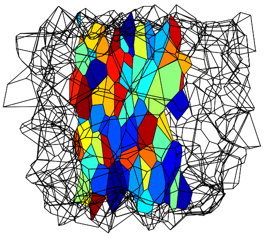

Voronoi diagram in two dimensions

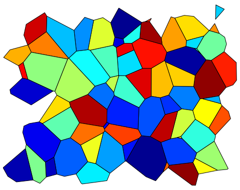

Voronoi diagram in two dimensions


Delaunay triangulation

In the following pictures, (a) shows a three-dimensional Voronoi structure viewed as a solid block, (b) the same structure rotated and with a cross section running through it and (c) the cross section as viewed perpendicular to the sectioning plane.

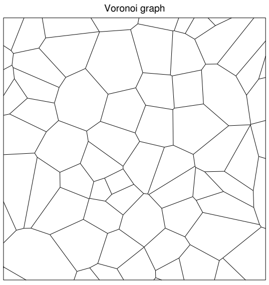

(a)


(b)

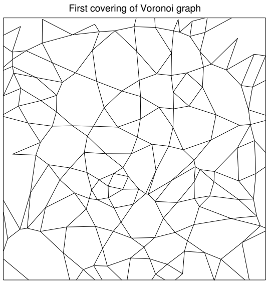

(c)

Let $C^n(x)$ be an $n^{th}$-order covering lattice of a tessellation $x$ in two dimensions. The following pictures show a two-dimensional Voronoi tessellation together with its covering lattices $C^1$ up to $C^3$.

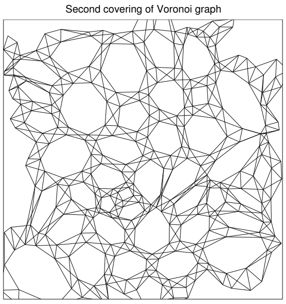
Voronoi tessellation ($x$)



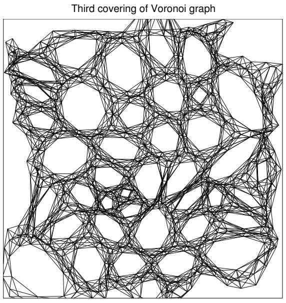
$C^1(x)$



$C^2(x)$



$C^3(x)$



The study of Voronoi tessellation has gained importance because it has been successfully used to describe many natural structures, for example it is linked to many cellular structures in nature and in Biotechnology. It has been used to study particle distributions in materials, the structure of porous media, and the structural properties of mixtures. It has been used to model anisotropic cellulose acetate and cellulose nitrate membrane, and to simulate animal cell cultures. The sand pattern formed by male mouth breader fishs, *Tilapia mossambica*, has been found to approximate a pattern of Voronoi polygons. Cluster analysis of Voronoi diagrams has been used in the analysis of clusters and scotomas in eyes in order to improve an early detection of visual field damage in glaucoma. In computer science it has been known under the name of *associative file searching* problem. Other applications are in finite element, pattern classification in statistics, dot pattern analysis, motion planning in robotics, etc. The justification for these many applications is that the statistics obtained from this set of structures compare nicely with measurements made from physical structures. Examples of such measurements are the Scanning Electro Micrograph of membranes and the pencil beam survey of the universe.

Because Voronoi structure comes up in wide and various topics, it has been known by various names. It has been called *Wigner-Seitz polygons* after Wigner and Seitz who first used it in metallurgy, *Blum's transform* after Blum who used it in visual science, *Dirichlet* and *Voronoi* tessellations after Voronoi and Dirichlet who pioneered and introduced it. At atomic level in the study of complex alloy structure it has been called *domain of an atom.* Other names include a *cell model*, a *plant polygon*, an $S$ polygon, a *Thiessen* polygon, and *Wirkungsbereich* (sphere or radius of operation). Thiessen used it to predict rainfall average of a region in meteorology. *Plant polygon* and *area potentially availble* were the terms given to area for an individual tree in a forest. The name $S$ polygon or $S$ mosaic sometimes arises in the field of ecology.

One way to mathematically describe a Voronoi region is

$$\Pi_i = \{x \mid d(x, x_i) < d(x, x_j) \; \forall j \neq i\}$$

where $x_1, x_2, x_3, \ldots$ are the coordinates of the nuclei with a distribution $\Phi$ of a countable set of nuclei $\{x_i\}$ in $\Re^d$ and $d(x, y)$ the Euclidian distance between $x$ and $y$.

Dual to the Voronoi tessellation is the Delaunay tessellation which can be thought of as the neighbouring relationship among Voronoi regions. Delaunay (Boris Nikolaevitch Delone, 1890–1980) was the person who redefined such relationship using a concept of empty spheres which is computationally useful and gave rise to the construction method of Voronoi tessellation by mean of Delaunay triangulation.

Some more on terminology. In 1980, B. Grünbaum and G. C. Shephard called the regions *plesiohedra* to comply with a fact that they belong to a set of regions called *stereohedra* whose congruent copies fill three dimensional space without overlap except possibly at their boundaries.

## On the reduction of positive quadratic form with three indeterminate integers.

([Lecture in physical- mathematical class meeting of the academy, on $31^{st} July, 1848$ †]), [by G. L. Dirichlet]

[translated by K N Tiyapan]

It is well known that Lagrange had pointed out for the first time that every binary quadratic form reduces, ie. can transform into another equivalent one the coefficients of which satisfy certain inequality conditions, and at the same time had proven that in every class of positive forms there always exists only one such form, so that in this case the various values of a given determinant corresponding to reduced forms can serve as the representatives of the different classes. Later on after in the "Disquisitiones arithmaticae" the ternary form were looked at from a general point of view did it become necessary for the further development of this theory to extend the study for the positive binary forms by Lagrange to the ternary ones, ie. to find out such inequality conditions between the coefficients that would satisfy one and only form in all classes.

This expansion linked with great difficulties is achieved by *Seeber* in a work specifically devoted to the positive ternary forms, the principal contents of which settles it and which *Gauss* characterises in a most interesting announcement † as follows:

> *We must do full justice to the spirit of the thoroughness by which these facing ‡ us have gone through, and when we for all that have to feel sorry that a great and perhaps much discouragingly complicated nature is attached to it, here the solution of the problem takes 41 pages and the proof 91 pages, thus we will see this by no means as a respected criticism. If a difficult problem or theorem to solve or to prove exists, then the first seeming idea is always to be recognised as a step that a solution or a proof has been found after all, and the question whether this were not of an easier and simpler way would be possible as long as in so doing such a futile question is not considered as of practicability. Therefore we look upon it as untimely to dwell on this question.*

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† From this treatise an excerpt has already been given in the monthly report, in which the principle of this new treatment of the positive ternary form, the examinations of successive minima, is outlined and the proof of the first one of the two results from Seeber after this principle completely carried out (page 21 of the second Volume of this issue of G. Lejeune Dirichlet's works.)

† Crelle's Journal, V. 20, p. 312

The great complication of Seeberian method has for a longtime stimulated me to set up the theory of reduced ternary forms by a simpler method. As I now allow myself to communicate to the class the result of my effort directed towards this, I think in the interest of briefness and, if I could say so, of the lucidity of the presentation, to have to abide by the geometrical form, in which I have conducted the investigation to which I have laid down as basis the noteworthy relations which occur among the quadratics with two or three elements and with known spatial forms. I begin with the explanation of the outline already given by Gauss in the mentioned announcement on these relations.

## §1

The ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy = \varphi, \tag{1}$$

in which we regard $x, y, z$ as first, second, third element, called positive when $\varphi$ does not become negative for real values of these elements; in one such form the coefficients:

$$a, b, c$$

are always positive, while the coefficient combinations:

$$a'^2 - bc, b'^2 - ac, c'^2 - ab, aa'^2 + bb'^2 + cc'^2 - abc - 2a'b'c' = -D, \tag{2}$$

the last $-D$ of which is called the determinant of the form, are negative. † Owing to these conditions there are always three through the equations:

$$\cos\lambda = \frac{a'}{\sqrt{bc}}, \cos\mu = \frac{b'}{\sqrt{ac}}, \cos\nu = \frac{c'}{\sqrt{ab}}$$

fully determined acute or obtuse angles $\lambda, \mu, \nu$, from which a three-edged corner can be built, here the condition necessary to this:

$$\cos^2\lambda + \cos^2\mu + \cos^2\nu - 2\cos\lambda\cos\mu\cos\nu < 1$$

with $D > 0$ coincided. Here nevertheless with the same angles $\lambda, \mu, \nu$ two corners symmetrical to each other could be built, therefore we will agree to always choose the corner from these two, with which the edges, as they lie opposite to these angles in sequence, follow one another from left to right with regard to a straight line directed from the vertex 0 to the inside of the corner which can be thought of as going upward. If we now look at the three edges as the positive axes of a coordinate system we could connect the entire infinite space with our form in which we view the product $x\sqrt{a}, y\sqrt{b}, z\sqrt{c}$ as the coordinates of an arbitrary point of the space, and then $\varphi$ expresses the square of the distance of this point from the vertex, or more general still the square of the distance of two points, the correspondent coordinates of which have those products to differences.

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† Disquisitiones arithmeticae, art. 271

If one establishes now with three new indeterminate elements $x', y', z'$ the linear expressions:

$$x = \alpha x' + \beta y' + \gamma z', y = \alpha' x' + \beta' y' + \gamma' z', z = \alpha'' x' + \beta'' y' + \gamma'' z', \quad (3)$$

of which only one restriction shall take place, that the determinant set up from the 9 coefficients $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma''$:

$$\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma'' = E \quad (4)$$

is not zero, then $\varphi$ changes into a new form $\varphi'$, with regard to which all correspondings shall be indicated with the accented alphabets. If one lets the new form again correspond to an infinite space, then through it two infinite spaces connect point for point with each other, while every two points correspond to each other when in the expressions of their coordinates:

$$x\sqrt{a}, y\sqrt{b}, z\sqrt{c}; \;\; x'\sqrt{a'}, y'\sqrt{b'}, z'\sqrt{c'}$$

the elements $x, y, z$ and $x', y', z'$ are linked with one another through the equation (3). If the expressions just written are the coordinate differences for two pairs of corresponding points, then apparently the same relation among $x, y, \ldots$ still holds, out of which from the above and as a result of $\varphi = \varphi'$ it follows immediately that the distance of every two points of a space is equal to the distance of corresponding ones of another. The two spaces connected with each other are therefore either congruent or symmetrical, ie. they can, while the beginning points 0 and $0'$ are laid on each other, come to such a position that either every point falls on its correspooning one or on the opposite point of the latter, when we call for short opposite points two points of the same space which lie from the beginning point at the same distance and in the opposite direction. In order to decide which of these two cases takes place, one has lines to draw in the one space from the vertex to three arbitrary points, and then to investigate whether the straight lines drawn in the other from its vertex to the corresponding points present a corresponding series or the opposite one. If one takes for example in the second space the lines from the points with the coordinates:

$$\sqrt{a'}, 0, 0;\ 0, \sqrt{b'}, 0;\ 0, 0, \sqrt{c'}$$

drawn, lines falling on the positive axes of the second space, then these follow one another from the agreement dealt with above from right to left. For the corresponding points in the first space one has the coordinates:

$$\alpha\sqrt{a'}, \alpha'\sqrt{a'}, \alpha''\sqrt{a'};\ \beta\sqrt{b'}, \beta'\sqrt{b'}, \beta''\sqrt{b'};\ \gamma\sqrt{c'}, \gamma'\sqrt{c'}, \gamma''\sqrt{c'}.$$

In order to determine whether the lines directed to these points follow one another from left to right, ie. as the axes of the first space, or follow in the reverse order, one can make use of the theorem which is known or easily derivable from known properties ‡, from which the straight lines drawn to the three points $(\xi, \eta, \zeta), (\xi', \eta', \zeta'), (\xi'', \eta'', \zeta'')$ present the same series as the axes of $\xi, \eta, \zeta$ or the opposite one, according to the determinant built from the 9 coordinates, when one gives the term $\xi\eta'\zeta''$ in it the positive sign, is positive or negative. For our case this determinant becomes $E\sqrt{a'b'c'}$; therefore congruence symmetry holds according as $E$ is positive or negative.

Til now the elements $x, y, z$ had arbitrary values. If we let them now only further mean integers, then instead of the integral space we have an infinite system of points parallelly arranged, ie. a point system of which through the intersections of three lines parallel equidistant planes would be created. If we assume now further that the substitution coefficients $\alpha, \beta, \gamma$ are also integers and $E$ has the values $\pm 1$, then every integral combination $x, y, z$ would represent an integral combination $x', x', z'$ and vice versa. The parallelepipedal systems thus connected with one another would as a result coincides with the other or with the opposite points of the latter. Yet, here the opposite points of points of one such system again make the same system, the two cases are not different from each other, and this becomes evident also from the circumstance that $\varphi'$ remains unchanged when one takes $\alpha, \beta, \gamma$ with opposite signs through which $E$ changes into $-E$. The two systems are therefore always congruent, and one sees that systems which correspond to two equivalent ternary forms $\varphi$ and $\varphi'$ are the same spatial structure in two different patterns. Conversely equivalent forms represent any two different parallelepipedal patterns of the same system. If one takes namely any one point of the system as the common starting point, then one has between the coordinates relation to the two axis systems and therefore also between the elements $x, y, z, x', y', z'$ proportional to them linear equations without constant term, ie. equations of the form (3), and here from our supposition, when $x, y, z$ are integers, $x', y', z'$ must also have the same characteristic and vice versa, therefore it follows that $\alpha, \beta, \gamma, \ldots$ are likewise integers and that $E = \pm 1$. On the other hand one has for the homogeneous entire values of the elements the equation $\varphi = \varphi'$, which accordingly also identically takes place, q.e.d.

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‡ Disquisitiones generales circa superficies curvas auctore, C. F. Gauss §2. VII

Similar interrelations occur between a positive binary form:

$$lx^2 + 2mxy + ny^2$$

and a system of points parallelogrammatically arranged. One takes here two axes leant against each other under the angle $\theta$ determined through the equation $m = \sqrt{ln} \cos \theta$, while one always invariably proceed with the discrimination of these axes and for example chooses the second on the left-hand side of the first one, after a fixed side of the plane is denoted as the higher one and $x\sqrt{l}, y\sqrt{l}$ viewed as coordinates, one would obtain a system of points completely determined through the quadratic form, which could be considered as the intersection of two series of equidistant parallel lines. If then between two forms the so-called proper equivalence takes place, so that $\alpha\delta - \beta\gamma$ in the substitution equations $x = \alpha x' + \beta y', y = \gamma x' + \delta y'$ is equal to the positive unity, then the corresponding systems can be brought to the coincidence through movement in the plane, while in the other case where $\alpha\delta - \beta\gamma = -1$, to say in general, one of the systems must be shifted for this purpose.

## §2

After we have established in the foregoing the connection between the quadratic forms and certain geometrical patterns, there are a few further properties of these patterns to develop, whereby we would for short call a system of points arranged parallelogrammatically or parallelepipedally a *system of second* or *third order*, and infinite series of equidistant points in straight line a *system of first order*.

It seems that the common character of all three types of the system consists in that when such a system is brought into another position through a movement without rotation, which we wish to know a displacement of, that a point of it changes into the position occupied by another in the beginning, the same happens for all points, therefore that the system in its new position fully coincides with the system in the original one. It can be easily proved that the movability just discussed completely characterises all three types of the systems, and that a system endowed with this characteristic, when it lies in a plane and contains three points not lying in a straight line such that finally a system contains points at least four of which are not found in a plane, will be respectively a system of first, second or third order.

If one has for example a system of points which lie all together in the same straight line, and $a$ and $a'$ are two adjacent points of it, then through a displacement through which $a$ gets to $a'$, $a'$ would get to $a''$ which is as far from $a'$ as $a'$ is far apart from $a$; the point $a''$ therefore also belongs to the system, and the system has no point between $a'$ and $a''$, here one such point would be known before the movement between $a$ and $a'$. Here this inspection could be pursued for both sides in the indeterminate, therefore the assertions is proved.

Now let two adjacent points $a$ and $a'$ be in a planar system with the characteristic feature of movability, so that no point of the system is found in the line $aa'$ between $a$ and $a'$. Here through the displacement from $a$ to $a'$ the infinite straight line $aa'$ moves along by itself, therefore it follows that the entire points of the system in this straight line makes up a system of first order $\ldots {}''a'aaa'a'' \ldots$. Here then from the assumption the system still has at least one point outside this straight line, therefore let $b$ be one of the points closest to this straight line. If now enters a displacement through which $a$ gets to $b$, then the system of first order changes into the new position $\ldots {}''b'bbb'b'' \ldots$ and belong in this position to the original system; it is immediately clear that a point of the system can neither be found among the points $\ldots, {}''b, {}'b, b, b', b'', \ldots$ nor among the lines $\ldots {}'bbb' \ldots, \ldots {}'aaa' \ldots$. If one concludes directly, one sees that the entire system can be parallelogrammatically aranged, and that one can choose $aa'b'b$ for a basic parallelogram of it. We further add that through the given construction apparently all parallelogrammatical patterns of which the system is capable could be obtained. It follows from this that the choice of $a'$ up to the obviously necessary restriction that no point lies between $a$ and $a'$ is totally arbitrary, and that $b$ can be taken arbitrarily in the nearest parallel line.

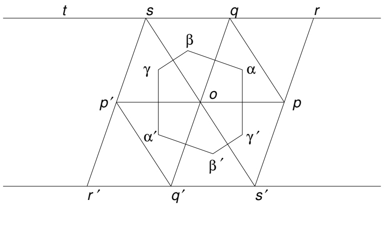
One has finally a system with the characteristic feature of the movability which contains points at least four of which are not lying in the same plane, therefore one lay a plane through any three points of this system not lying in a straight line. Here through any parallel displacement effected with this plane this is moved into itself, consequently points found in this plane build a system of second order from the previous system. As a result one has partitioned this system parallelogrammatically somehow or other, one takes one of the remaining points of the spatial system which lie closest to the plane, and administer a displacement to the system through which an arbitrary point in the plane comes to the point chosen well outside that plane. Through repeated application of this [displacement] and through the movement opposite to it one apparently obtains a parallelepipedal pattern of the given system, and it is immediately clear that the construction specified has the due generality, here the choice of the first plane, the pattern of the system of second order in this plane and finally the choice of points in the neighbouring plane can happen at will.

In the end of this paragraph we will point out further that, as one also partition the same system of second or third order, the parallelogram or parallelepiped lying at the basis of the respective partitioning always retains the same capacity, the geometrical consequence of the sentence is that equivalent forms have the same determinants. If one imagines namely in the plane of a system of second order a line returning to itself, for example a circle line, designates with $z$ the surface area enclosed by it and with $s$ the number of points in the inside of the line, in the course of which it makes no difference whether one wants to include the points on the periphery or not, then obviously the quotient $\frac{z}{s}$ has with growing radius the capacity of a basic parallelogram to the boundary, from which, here $s$ and $z$ are independent of the type of the pattern, the theorem for systems of second order becomes evident. Totally in the same fashion follows the soundness of the assertions for spatial systems.

## §3

We will now point out that a system of second order would always admit partitioning by a basic parallelogram, the sides of which are not larger than its diagonals.

I. Let $o$ be an arbitrary point of the system. The remaining points of this system always lie pairwise in the same distance and opposite direction from $o$. Now let $p$ be one of the points of the pairs, for which the distance from $o$ is smaller than for every other pair. The same smallest distance holds for more than one pair, therefore one would choose $p$ at will in one of these. The given system con-



sists of an infinite quantity of systems of first order congruent among one another and of the same distance, one of which is that for which $o$ and $p$ belong. In one of the two adjacent to this latter one, one takes the point $q$ which is next to $o$, or, supposing the same shortest distance should occur for two points, arbitrarily takes one of the two. The parallelogram $poqr$ thus obtained has the desired property, here in accordance with the construction $op \leqq oq$, $oq \leqq or$, $oq \leqq os = pq$. A basic parallelogram which satisfies these conditions shall be called a reduced one.

II. We have now the relation between one such parallelogram and the planar system in which it belongs to establish. If $poqr$ is a reduced parallelogram we would be able to, without breaking the generality, assume the angle $poq$ as not obtuse, here in the opposite case the angle at $o$ for the parallelogram adjacent to the same structure is an acute one, and likewise we could assume $op \leqq oq$. Thereupon $or > oq$ is apparent, and we have only the condition $pq \geqq oq$ still to consider. If this is supposed and if we put for the reduction $op = \sqrt{l}$, $oq = \sqrt{n}$, so that consequently $l \leqq n$ the connection of our parallelogram to the entire point system would possibly be described to the effect that the minimum of the distance of any point of the system from $o$ is equal to $\sqrt{l}$, and that after one has chosen a point at this distance, in all distances still remaining, ie. outside the straight line drawn from $o$ to the former one, the second minimum is equal to $\sqrt{n}$. The precisely stated holds true all in general; what we now add, that namely the first minimum only occurs for the point $p$ (when we always only choose one of two opposite points), the second only for $q$, holds true with the following exceptions:

1. If $op < oq$, $oq = pq = os$, then the first minimum takes place only for $p$, the second for $q$ and $s$.

2. If $op = oq$, $oq < pq = os$, then the minima are equal, and one can exchange $p$ and $q$ with one another.

3. If finally $op = oq = pq = os$, then one can choose one of the points $p, q, s$ as first point and then one of the remainings as second one.

In order to demonstrate the precisely asserted, we have obviously, the opposite points are always equally far from $o$, only to point out that $q$ lies closer to $o$, *firstly* than all remaining points in the straight line $sqr$, with exception of the point $s$, the distance from $o$ of which according to the assumption is equal to $os = pq \geqq oq$, and *secondly* than all points of the subsequent parallel lines.

Here $pq \geqq op$, $pq \geqq oq$ and the angle $poq$ is not obtuse, therefore the triangle $opq$ and consequently also the $oqs$ congruent with it has no obtuse angle; therefore the perpendicular dropped from $o$ on $qs$ lies between $s$ and $q$ (inclusive), with which

one sets:

$$\cos poq = \frac{m}{\sqrt{ln}},$$

where consequently $m$ is not negative, therefore one has:

$$\overline{pq}^2 = l - 2m + n \geqq \overline{oq}^2 = n,$$

and hence:

$$2m \leqq l, 2m \leqq n, 4m^2 \leqq ln.$$

If one further sets the square of the height of our parallelogram ($op = \sqrt{l}$ regarded as base line) equals $k$, one obtains for the square $\Delta$ of its volume:

$$\Delta = lk = ln - m^2 \geqq \frac{3}{4}ln,$$

and hence:

$$\sqrt{k} \geqq \frac{1}{2}\sqrt{3n}.$$

According to this the second line is at least $\sqrt{3n} = oq\sqrt{3}$ away, and the second point is also established.

III. Here the successive minima $\sqrt{l}$, $\sqrt{n}$ are decided through the system as such and are independent of any fixed pattern and on the other hand, as we have just seen, correspond in quantity with the sides of the reduced parallelogram, therefore one sees that when the system permits various patterns of this fashion, the sides of the reduced parallelograms will always contain the value $\sqrt{l}$ and $\sqrt{n}$. One would essentially obtain as a result all possible basic parallelograms if one draws lines from $o$ to all adjacent points (always with exemption of the opposite points) and then takes the nearest or the two nearest points in one of the respective nearest parallel lines; and here from the definitively demonstrated (II) this nearest or these nearest points lie closer to $o$ than all points of the subsequent parallel lines, therefore one can see from the condition that the second points are to be taken in the first parallel line. Therefore all possible patterns would be produced if one successively connect $o$ with all point pairs for which the successive minima take place, from which at once follows with consideration from (II) that in general and in the second one of the singular cases obtained then there is only one such pattern, in the first and third exception case however there are respectively two and three patterns of the systems.

In our present reference the precisely obtained singular cases correspond with the suppositions $2m = l < n$, $2m < l = n$, $2m = l = n$.

### §4

We have so far only dealt with properties of the geometrical structures which is to be looked at from the theory of forms as the constructive representation of well known theorems and are already indicated in the article cited in the introduction. It is now to solve another problem of another kind, the problem namely when a system of second order is given and a fixed point $o$ of it is examined, to determine the part of the plane every point of which lies nearer to $o$ than to any other points of the system. Here the condition that a point does not lie farther from $o$ than from any other $v$, therein consists that the point with $o$ on the same side of the perpendicular drawn up in the middle of $ov$, so we would consequently have $o$ to combine with all remaining points of the system and the convex polygon built from all corresponding perpendiculars to construct. But from these perpendiculars in infinite quantity only a limited number comes into question, while the remaining ones do not meet the polygon determined by it. We abide by all suppositions attended to, so that consequently $op \leqq oq$ in the reduced parallelogram $(poqr)$, the angle $poq$ is not obtuse and $opq$, $oqp$ are acute. This supposes, it is easy to understand, that one has only the six vertices $p, q, s, p', q', s'$ of the four parallelograms meeting at $o$ to take into consideration, and that the perpendiculars corresponding to $s$ and $s'$ and the building diagram in the particular case, when $poq$ is a right angle, only touch, which then the same happens for the perpendiculars corresponding to $r$ and $r'$. If one draw the straight line $pq$, $os$, $p'q'$, $os'$, one obtains the congruent triangles:

$$poq, qos, sop', p'oq', q'os', s'op.$$

If one consider only the points $p, q, s, p', q', s'$, one has to draw a perpendicular in the middle of the straight lines going from $o$ to these points, ie the same construction to make as when one wished t find the middle point of circumscribed circles for the designated triangles. Here no obtuse angle is found in the triangles, therefore each two successive perpendiculars not outside the corresponding triangle intersect. One obtains therefore the hexagon $\alpha\beta\gamma\alpha'\beta', \gamma'$ with the centre $o$ and equal opposite angles and sides as the space, inside of which every point is less far apart from $o$ than from one of the points $p, q, s, p', q', s'$, and one is easily convinced that, with exception of $r$ and $r'$, the perpendiculars corresponding to the remaining points do not meet our hexagon. This requires, as a result of symmetry, only for the points in and above the line $pop'$ to be established. For the former ones it is clear; for the latter ones it would hence appear that their distance from $o$ is larger than the diameter of the circle traced around the hexagon. If one designate the square of its radius $\rho$, then:

$$4\rho\Delta = ln(l - 2m + n),$$

from which as a result of $2m \leqq l, 2m \leqq n, \Delta \geqq \frac{3}{4}ln$, it follows:

$$4\rho \leqq \frac{4}{3}(l - 2m + n) \leqq \frac{8}{3}n.$$

Here now for the points of the second and the subsequent parallel lines, as already remarked, the square of their distance from $o$ amounts to at least $3n$, therefore there still remain simply the points in $tsqr$ apart from $s, q, r$ to examine. From all of these none is closer to $o$ than $t$, for which the square of the distance is equal to $4l - 4m + n$, and that this is larger than $4\rho$, one immediately sees when one multiplies with $\Delta$ and then looks at the inequalities $2m \leqq l \leqq n$. As for the point $r$, one is also convinced by the same manner that the square of its distance from $o$ is equal to $l + 2m + n > 4\rho$, the only case excluded, where $m = 0$, in which the corresponding perpendicular touches. It is thus demonstrated that every point in the inside of the hexagon $\alpha\beta\gamma\alpha'\beta'\gamma'$, and only one such hexagon, lies closer to the point $o$ than any other of the system. On any side the distance from $o$ would be equal to the distance from a second point, which for example for $\alpha\beta$ is the point $q$, and every vertex of the diagram is of the same distance from $o$ and another two other points of the the system. The latter statement undergoes a modification only in the special cases when the angle $poq$ is a right angle; thereupon $\beta$ and $\gamma$ as well as $\beta'$ and $\gamma'$ coincide, and the hexagon turns into a rectangle, of which the corner from $o$ and another three other points of the system are equally far apart.

It goes without saying that one will always obtain the same hexagon whose reduced parallelogram one also lay as foundation of the construction in the singular cases, where more than one exists, just as also that the hexagon or quadrangle corresponding to all the points of the system are congruent and cover the whole plane of it.

We notice further that, as one is easily convinced, the expression:

$$\rho = \frac{ln(l - 2m + n)}{4(ln - m^2)}$$

decreases when one therein, assuming $l$ and $n$ constant, allows $m$ to grow from zero up to its limit $\frac{1}{2}l$, so that consequently:

$$\rho \leqq \frac{1}{4}(l+n) \leqq \frac{1}{2}n. \tag{1}$$

Also in addition the following inequality takes place:

$$2\Delta(n-\rho) \geqq ln^2, \tag{2}$$

the soundness of which is immediately evident when one multiplies with 2, moves everything to one side and then applies $\Delta = ln - m^2$, $4\Delta\rho = ln(l - 2m + n)$, by the mean of which it changes into $ln(l - 2m) + 2mn(n-l) \geqq 0$.

### §5

We come now to our true topic and have to prove that every system of third order can be arranged according to a parallelepiped whose faces are reduced parallelograms and whose edges, from which every four are equal to one another, do not exceed their diagonals.

After one has fixed an arbitrary point (0) of the systems, one would choose in pairs of opposite points for which the distance from (0) is a minimum, or when the minimum of the distance exists for several pairs, would arbitrarily choose a point (1) in one of these pairs. From all points outside the straight line (01) one would again choose one of the two nearest (2), through which again the selection under several pairs, for which the same shortest distance takes place, can be arbitrarily made. Here in the whole system, with exception of the points in (01), no point lies closer to (0) than to (2), so the same is valid also for the plane (102), and (102) is a reduced parallelogram for the system which contains this plane (§3, III). One now takes in one of the two nearest parallel planes the point which is closest to (0) or, when the minimum occurs for more than one, one of the nearest ones and connect (0) with the chosen point (3), therefore the parallelepiped would with the edges (01), (02), (03), as is easy to see, suffice the requirement. Next from the construction it follows: (01) $\leqq$ (02) $\leqq$ (03). Here for the bases of the parallelepiped (we would always indicate as such each face opposite to one another in which are found edges two of which do not exceed the third one in size, and the term side faces apply to the four remaining ones) it is already proven that they are reduced, therefore we have in virtue of the precisely noted doubled inequality only to point out further that the four diagonals of the side faces, just as the four diagonals of the body, are not smaller than (03). Now the eight diagonals mentioned above will agree, as one immediately sees, in size with the eight connecting lines which could be drawn from (0) to the eight points lying around (3) in the plane of the higher bases if we indicate this way for convenience the eight vertices of the four parallelograms meeting at (3). The fact that from the afore-mentioned connection lines none is smaller than (03) follows from the condition by which (3) is being chosen.

After we have convinced ourselves that a system of third order can always be partitioned by a reduced parallelepiped, we now have to establish the relation between such parallelepiped and the system and particularly to compare the distance of the point of systems from (0) with one another. We set $(01) = \sqrt{a}, (02) = \sqrt{b}, (03) = \sqrt{c}$ and always hold fast the assumption $a \leqq b \leqq c$.

1. In the plane of the base the conditions discused above (§3, II) occurs, so that consequently the successive minima of the distance are always $\sqrt{a}, \sqrt{b}$ in size, whereby then in the singular cases mentioned there there is an arbitrariness in the choice of the points.

2. We now look at the points outside the plane of the base underneath namely first of all the one in the plane of the base above. Here from the assumption that our parallelepiped is a reduced one, the line (03) is not longer than one of the straight lines drawn from (0) to the eight points lying around (3), so as a result the foot of the perpendicular dropped from (0) on to the plane of the base above would not be farther apart from (3) than from one of the eight points mentioned. This foot point therefore does not fall outside the hexagon or quadrangle constructed to belong to (3) in the last paragraph. Of those eight points can exceptionally, when the foot point falls on one side, one, or it could, when the foot point meets with a vertex, two (three, when the polygon becomes a rectangle,) lie equally close to the foot point as the point (3), while all remaining points of the plane are further apart from that foot point. It follows from this that the shortest distance (amounting to $\sqrt{c}$) from (0) to the a point in the base above is valid in general only for the point (3), but can exceptionally take place for one, two or even three other points.

3. For the consideration of the following parallel planes we have a boundary for the square $h$ of the perpendicular already mentioned to determine. Here the foot point of the perpendicular does not fall outside the hexagon which belongs to (3), therefore, when $\rho$ denotes the square of the radius of the circumscribed circle:

$$h \geqq c - \rho.$$

Now also from §4: $\rho \leqq \frac{1}{2}b \leqq \frac{1}{2}c$, consequently $h \geqq \frac{1}{2}c$. Here therefore the second parallel plane is at least $\sqrt{2c}$ away, therefore there is over the higher base only points, the distance from (0) of which is greater than $\sqrt{c}$.

If one summarises what has been said, one will see that the minimum of the distance for the entire system has the value $\sqrt{a}$, that, after a point is chosen at this distance, the minimum in the still remaining directions amounts to $\sqrt{b}$, and that finally after the second point is also fixed, for all points outside the plane, which is determined through (0) and the first two points, the smallest distance from (0) is reduced to $\sqrt{c}$. If the successive minima $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are also always completely determined in quatity, the same minimum in local relation will not be true without several exceptions which are easy to specify. If for example $a \leqq b$, $b < c$, the first two points are to be chosen in the lower base, whereby the singular cases mentioned in §3, II could occur, while the third point lies in the higher base, has a fixed position there in general, in singular cases however can occupy two, three or four different places. One ever so easily overlook that varieties in the other two cases, where $a < b = c$ or $a = b = c$, could happen.

Here from the assumption of a reduced parallelepiped with the edges $\sqrt{a} \leqq \sqrt{b} \leqq \sqrt{c}$ the length of these edges have yielded themselves as the successive minima of the system, thus it immediately follows that when several reduced parallelepiped exist from which the system can be arranged, these all become in agreement with one another with regard to the lengths of their edges, and it could also be easily pointed out that three of the lines directed from (0) to points of the systems of the lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$ when they only do not lie in the plane, are always the edges of a reduced parallelepiped. It requires therefore only the easy consideration already applied in a similar case (§3, III). Here after this the entire reduced parallelepipeds would be obtained when one construct the successive minima of every possible types, therefore it becomes evident that when this can happen in only one way (to which we also consider the case where, with the equation of two of the quantities $\sqrt{a}, \sqrt{b}, \sqrt{c}$ or with the equation of all three, the three lines are locally completely determined and only an exchange between two or all three can occur) that spatial system would allow only one pattern from a reduced parallelepiped. In all other cases there are several such patterns, the parallelepipeds of which form the basis, which could be either all different from one another or only different in part or even could be all congruent to one another. (Similarly in the two singular cases of a system of second order mentioned above the reduced parallelograms underlying the two or three various patterns were congruent to one another.)

To the determination of the question whether a system of third order permits only one or more than one pattern from a reduced parallelepiped, it would consequently only need the knowledge of a single pattern of the system, and the first case would always and exclusively take place when the reduced parallelepiped given through this pattern is of such a property that all lines which can not be exceeded by others actually exceed this parallelepiped, that is when all diagonals of the faces are larger than their sides and all diagonals of the parallelepiped are similarly larger than the edges of the bodies.

## §6

As we now apply the results of the last paragraph to the ternary form, shall the uniformity be assumed because of and in order to avoid pointless differentiation, that one has given every ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy \tag{1}$$

through transposition or change of sign of indeterminate elements, as a result of which the form does not belong to the same class, a single form, that firstly $a \leq b \leq c$ holds, that secondly under the coefficients $a', b', c'$, when not the case that all of them are nonzero and are negative, none has the negative sign, and thirdly, when $b = c$ holds, $c'$ apart from the sign not greater than $b'$, when $a = b$ holds, $b'$ not greater than $a'$, and lastly when $a = b = c$ holds, neither $c'$ greater than $b'$ nor $b'$ greater than $a'$ holds. As is easy to see these condition can only be satisfied in one way and their introduction gives the advantage that, as already without these conditions every ternary form corresponds with a completely determined parallelepiped, now to every parallelepiped also belongs an analytical expression the coefficients of which are also completely determined with regard to their sequence and their signs. This assumed, we mention the form (1) in which $a \leq b \leq c$ also holds, a reduced one, when it corresponds with a reduced parallelepiped. There the diagonal of the area must not be smaller than the sides themselves, so one has:

$$a \pm 2c' + b \geq b,\ a \pm 2b' + c \geq c,\ b \pm 2a' + c \geq c.$$

One sets $\sigma = -1$, where $a', b', c'$ are all three negative, otherwise $\sigma = 1$, so these conditions are synonymous with:

$$a \geq 2c'\sigma,\ a \geq 2b'\sigma,\ b \geq 2a'\sigma, \tag{2}$$

and only, when the equal sign holds, would one of the diagonals in the corresponding parallelogram be equal to a side. The conditions with regard to the diagonals of the parallelepiped result in:

$$a + b + c + 2a'\epsilon + 2b'\delta + 2c'\delta\epsilon \geq c \quad (\delta = \pm 1, \epsilon = \pm 1),$$

where the signs in $\delta = \pm$, $\epsilon = \pm 1$ are arbitrary. One look next at the case where none of the coefficients $a'$, $b'$, $c'$ is negative, and take into account the four sign combinations, as well as that, when $a$ and $b$ are equal to one another then $b' < a'$, so one immediately sees that our inequality is by itself capable of always meet the condition contained, and that the limiting case of the equation in which the diagonals of the edge $\sqrt{c}$ become equal, only once and only then can it occur, when one of the quantities $b'$, $c'$ is equals to zero and when at the same time of the conditions (2) the two quantities $b'$, $c'$ of which are relating to one another, as well as the one which contains $a'$, satisfies the limiting case of the equation. $a'$, $b'$, $c'$ are negative, then our inequality is always fulfilled that the limiting case can not take place, except when $\delta = \epsilon = 1$, so that the consequently the new condition is established:

$$a + b + 2a' + 2b' + 2c' \geq 0, \tag{3}$$

where again the lower sign relates to the equality between a diagonal and the edge $\sqrt{c}$.

The condition just developped (2) and, when $a'$, $b'$, $c'$ are negative, (3) above is are therefore fulfilled, that the inequality takes place in none of the inequalities of the limiting case, therefore in the class to which the form belongs it would not give a second one of these various ones with or without equality signs in the definition condition, here according to at the end of the last paragraph notice that the corresponding system of points can only be partitioned from a reduced parallelepiped. The matter stands differently when the upper signs do not take place in all conditions; there could then occur in the same class several reduced forms that can be derived from a given one. It is sufficient to demonstrate this for a main case. We choose for this the case where $b < c$.

Next one assumes $a > 2c'\sigma$, therefore the direction of the edge $\sqrt{c}$ can only be altered when there are namely in the plane of higher base still one or more points, the distance from the vertex of which amounts to $\sqrt{c}$. When $\xi, \eta$ are 1 the one such points corresponding values of the element so would, when the third edge depends on it, all the coefficients except $a', b'$ remain unchanged, these respectively change into $a' + c'\xi + b\eta$, $b' + a\xi + c'\eta$ as one is easily and almost convinced without calculation. Now from the specification made earlier are the values of $\xi, \eta$ which meet the condition, when $a = 2b'\sigma$:

$$\xi = -\sigma, \quad \eta = 0;$$

when $b = 2a'\sigma$:

$$\xi = 0, \quad \eta = -\sigma;$$

when simultaneously $a = 2b', b = 2a', c' = 0$:

$$\xi = -1, \quad \eta = -1;$$

and when $a', b', c'$ are negative and the equation $a+b+2a'+2b'+2c' = 0$ complied with:

$$\xi = 1, \quad \eta = 1.$$

Corresponding to these four assumptions one has consequently transformed $a', b'$ into:

$$a' - c'\sigma,\ b' - a\sigma (= -b');\ -a',\ \ b' - c'\sigma;\ -a',\ \ -b';\ a' + b + c',\ a + b' + c'.$$

From the third case and generally from the assumption $c' = 0$ one can foresee, here this represents a new form which afterwards one has undertaken in the same one the change of sign stated in the beginning of this paragraph, apparently with the form from which one has derived, become identical. In each of the three remaining assumptions one obtains from application of the necessary sign change a new reduced form belonging to the same class (provided that it does not coincide with the original one), and one obtains two such forms when two of our assumptions hold at the same time. With this then the specification of the form is brought to an end, here apparently the simultaneity of all three assumptions can not take place. If, always under the assumption $b < c$, $a = 2c'\sigma$, one would have, provided that $a < b$, the two edges rotated in the base, for $a = b$ the first edge can also pass into the original position of the two first edges and this new position or both of these new positions of the first two edges must be associated with all the directions of the third one, the original one not excluded.

One can thereby easily remove the inconvenience that in singular cases several reduced forms can be associated in the same class, and thereby take away the exception that for such singular cases in general definition one still include the known secondary conditions which can be proved when one for instance set up the demand that the last coefficient $c'$, provided that it is not fully fixed, maintains the smallest numerical value of which it is capable in the reduced forms of the class, and then likewise with regard to $b'$. For this to notify an example, we will examine it under the singular cases dealt with earlier where $b < c$, from the three conditions (2) none with the lower signs holds, however the three negative values $a', b', c'$ satisfy the equation:

$$a + b + 2a' + 2b' + 2c' = 0.$$

From the previous observations $c'$ is fixed, and there exists for this case only two reduced forms. $a'$ and $b'$ are the values of the forth and the fifth coefficients in one of them, therefore they are in the other $a' + b + c'$, $a + b' + c'$, or, here the last value are apparently positive, $c'$ is negative and consequently, in order for the sign specification to be sufficient, $z$ to transform into $-z$, rather $-(a' + b + c')$, $-(a + b' + c')$. As it is quite natural, these values suffice when one substitute them for $a', b'$, again the equation:

$$a + b + 2a' + 2b' + 2c' = 0,$$

and from them come the values $a', b'$ in the same manner, as they themselves are originated from $a', b'$. Here according to this the fifth coefficient admits only the two negative values $b'$ and $-(a + b' + c')$, their sum equals $-a - c'$, therefore one see that when one further add to the definition conditions:

$$-b' \leqq \frac{1}{2}(a + c'),$$

the class would contain only one reduced form.

While we conclude the essay, we will still from our principles derive a beautiful theorem, found by *Seeber* through induction and demonstrated by *Gauss* in the announcement already often quoted. From this theorem in a reduced form the production of the first three coefficients is not larger than the doubled absolute value of the determinant.

Here the absolute value of the determinant is equal to the square of the volume of the parallelepiped corresponding to the form, thus consequently, from the expression employed in §5, 3 the inequality to be proved:

$$abc \leqq 2\Delta h,$$

where $\Delta$ represents the square of the base. One sets:

$$c = b + t,$$

where consequently $t$ is not negative, draws off from the inequality obtained in §5, 3:

$$h \geqq c - \rho = b - \rho + t,$$

from which one has multiplied it with $2\Delta$, the equation:

$$abc = ab^2 + abt,$$

thus one obtains:

$$2\Delta h - abc \geqq 2\Delta(b - \rho) - ab^2 + (2\Delta - ab)t.$$

Here now from the inequality at the end of §4, $2\Delta(b - \rho) - ab^2$ is not negative and $2\Delta - ab \geqq \frac{1}{2}ab$ is positive, therefore the truth of the theorem becomes evident.

New applications of continuous parameters to the theory of the quadratic form

## First Memoir

# On some properties of the perfect positive quadratic forms

by Mr. Georges Voronoï in Warsaw

[*Journal für die reine und angewandte Mathematik*, V. 133, p. 97–178, 1908]

[translated by K N Tiyapan]

## Introduction

Hermite had introduced in the theory of numbers a new and fruitful principle, namely: being given a set $(x)$ of systems $(x_1, x_2, \ldots, x_n)$ for all the values of $x_1, x_2, \ldots, x_n$, one associates with the set $(x)$ a set $(R)$ composed of the domains in a manner such that by studying the set $(R)$ one studies at the same time the set $(x)$ .

Hermite has shown † numerous applications of the new principle to the generalisation of continuous fractions, to the study of algebraic units, etc.

The ideas of Hermite have been developped in the works of Mr.'s Zolotareff, Charve, Selling, Minkowski. ‡

I intend to publish a series of Mémoires in which I shall show new applications of the principle of Hermite to the various problems of the arithmetic theory of definite and indefinite quadratic forms.

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† *Hermite.* Extraits de lettres de M. Ch. Hermite à M. Jacobi sur differents objets de la théorie des nombres. [Excerpts from letters of Mr. Ch. Hermite to Mr. Jacobi on various subjects in the theory of numbers] (This Journal V. 40, p. 261)

*Hermite.* Sur l'Introduction des variables continues dans la théorie des nombres. [On the introduction of the continuous variables in the theory of numbers] (This Journal V. 41, p. 191)

*Hermite.* Sur la théorie des formes quadratiques. [On the theory of quadratic forms] (This Journal V. 47, p 313)

‡ *Zolotareff.* On an indeterminate equation of the third degree (Petersbourg, 1869, in Russian.)

*Zolotareff.* Theory of complex integers with applications to the integral calculus. (Petersbourg, 1874, in Russian.)

*Charve.* De la réduction des formes quadratiques ternaires positives et de leur application aux irrationelles de troisième degré. [Of the reduction of positive ternary quadratic forms and of their application to the irrationals of third detree] (Suppl. to V. IX of Annales Scientifiques de l'Ecole Normale Supérieure, 1880)

*Selling.* Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p. 143)

*Minkowski.* Geometrie der Zahlen. [Geometry of numbers] (Leipzig, 1896)

In this Mémoire, I study the properties of the minimum of positive quadratic forms and of their various representations by systems of integers.

*Hermite* has discovered an important property of the minimum $M$ of positive quadratic forms $\sum a_{ij}x_ix_j$ in $n$ variables and of the determinant $D$, namely:

$$M \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D},$$

and he has demonstrated numerous applications of this formula.

In a letter to Jacobi, Hermite has said §:

"That which precedes sufficiently indicates an infinity of other analogous consequences which, all, will depend on the difficult study of an exact limit of the minimum of any definite form. Thereupon I then form only one conjecture. My first studies in the case of a form in $n$ variables of the determinant $D$ have given me the limit $\left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D}$, I am inclined to presume, but without being able to demonstrate that the numerical coefficient $\left(\frac{4}{3}\right)^{\frac{n-1}{2}}$ has to be replaced by $\frac{2}{\sqrt[n]{n+1}}$"

Mr.'s Korkine and Zolotareff has under taken the study of the exact limit of the minimum of positive quadratic forms of the same determinant.

By indicating with $M(a_{ij})$ the minimum and with $D$ $(a_{ij})$ the determinant of the form $\sum a_{ij}x_ix_j$, one will have the minimum

$$\mathcal{M}(a_{ij}) = \frac{M(a_{ij})}{\sqrt[n]{D(a_{ij})}}$$

of a positive quadratic form with determinant 1.

By virtue of the theorem of Hermite the function $\mathcal{M}(a_{ij})$ verifies the inequality ¶

$$\mathcal{M}(a_{ij}) \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}},$$

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§ This Journal. V 40, p. 296

¶ Mr. Minkowski has demonstrated an upper limit of the function $\mathcal{M}(a_{ij})$

$$\mathcal{M}(a_{ij}) \leq n$$

much simpler than that from Hermite. (*Minkowski.* Über die positiven quadratischen Formen und über kettenbruchähnliche Algorithmen. [On the positive quadratic forms and on continued fraction algorithm] This Journal V. 107, p. 291) therefore it is bounded within the set $(f)$ of all the positive quadratic forms of real coefficients.

Mr.'s Korkine and Zolotareff have demonstrated † that the function $\mathcal{M}(a_{ij})$ possesses many maxima in the set $(f)$ which correspond to the various classes of equivalent positive quadratic forms.

The limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite in the letter to Jacobi (source cited) is only a maximum value of the function $\mathcal{M}(a_{ij})$.

The binary and ternary positive quadratic forms possess a single maximum which is therefore, in this case, the exact limit of values of the function $\mathcal{M}(a_{ij})$.

Reckoning from the number of variables $n \geq 4$, one meets many maxima of the function $\mathcal{M}(a_{ij})$.

Mr.'s Korkine and Zolotareff have found many values of various maxima of the function $\mathcal{M}(a_{ij})$ which exceed the limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite, but do not exceed the limit 2.

The study of the exact limit of the minimum of positive quadratic forms of the equal determinant comes down, after Mr.'s Korkine and Zolotareff, to the study of all the various classes of positive quadratic forms to which correspond the maximum values of the function $\mathcal{M}(a_{ij})$.

The maximum maximorum of values of the function $\mathcal{M}(a_{ij})$ is the largest value of the function $\mathcal{M}(a_{ij})$ which presents a numerical function as $\mu(n)$.

Mr.'s Korkine and Zolotareff have determined the following values of the function $\mu(n)$:

$$\mu(2) = \sqrt{\frac{4}{3}},\ \mu(3) = \sqrt[3]{2},\ \mu(4) = \sqrt[4]{4},\ \mu(5) = \sqrt[5]{8},.$$

They have called extreme the quadratic forms which yield to the function $\mathcal{M}(a_{ij})$ a maximum value.

The extreme quadratic forms enjoy an important property, namely:

*I. Any extreme quadratic form is determined by the value of its minimum and by all the representations of the minimum.*

Mr.'s Korkine and Zolotareff have determined all the classes of extreme forms in 2, 3, 4 and 5 vertices.

By studying these extreme forms, I have observed that they are all well defined by the property (I). There is only reckoning from positive forms in six variable which I have encountered positive quadratic forms which enjoyed the property (I) and are not of extreme forms.

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† *Korkine* and *Zolotareff.* Sur les formes quadratiques. [On the quadratic forms] Mathematische Annalen, V. VI, p. 366 and V. XI, p. 242

I call "perfect" any positive quadratic form which enjoys the property (I).

I demonstrate that the set of all the perfect forms in $n$ variables can be divided into classes the number of which is finite.

All extreme form being, by virtue of the property I, a perfect form, it results in that the function $\mu(n)$ presents the maximum of values of the function $\mathcal{M}(a_{ij})$ which correspond to the various classes of perfect forms.

I have established an algorithm for the search of various perfect forms by introducing a definition of contiguous perfect forms.

To that effect, I make correspond to the set $(\varphi)$ of all the perfect forms in $n$ variables a set $(R)$ of domains in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

The set $(R)$ of domains in $\frac{n(n+1)}{2}$ dimensions presents a partition of the set $(f)$ of all the positive quadratic forms in $n$ variables.

Each domain $R$ possesses in the set $(R)$ a contiguous domain which is well determined by any one face in $\frac{n(n+1)}{2} - 1$ dimensions of the domain $R$.

I demonstrate that the domain $R$ corresponding to the perfect form $\varphi(x_1, x_2, \dots, x_n)$ being determined by the linear inequalities

$$\sum p_{ij}^{(k)} a_{ij} \geq 0, \quad (k = 1, 2, \dots, \sigma)$$

one will have $\sigma$ perfect forms defined by the equalities

$$\varphi_k(x_1, x_2, \dots, x_n) = \varphi(x_1, x_2, \dots, x_n) + \rho_k \Psi_k(x_1, x_2, \dots, x_n), \quad (1)$$
$$(k = 1, 2, \dots, \sigma)$$

where

$$\Psi_k(x_1, x_2, \dots, x_n) = \sum p_{ij}^{(k)} x_i x_j,$$

provided that the positive parameter $\rho_k$ $(k = 1, 2, \dots, \sigma)$ presents the smallest value of the function

$$\frac{\phi(x_1, x_2, \dots, x_n) - M}{-\Psi(x_1, x_2, \dots, x_n)}$$

where $\Psi(x_1, x_2, \dots, x_n) < 0$ and $M$ is minimum of the form $\phi(x_1, x_2, \dots, x_n)$.

I call "contiguous to the perfect form $\phi(x_1, x_2, \dots, x_n)$" the perfect forms (1).

Any substitution in integral coefficients and with determinant $\pm 1$ belonging to the group $g$ of substitutions which do not change the form $\phi$ permute only the forms (1). One can, therefore, divide the forms (1) into classes of equivalent forms with the help of substitutions of the group $g$. By choosing one form in each class, one will have a system of perfect forms contiguous to the perfect form $\phi$ which can replace the system (1).

By proceeding in this manner, one can obtain a system complete of representatives of various classes of perfect forms.

The corresponding domains will form complete system of representatives of various classes of the set $(R)$.

I have remarked that a similar system

$$R, R_1, R_2, \dots, R_{\tau-1} \tag{2}$$

of domains of the set $(R)$ can serve towards the reduction of positive quadratic forms.

I call reduced any positive quadratic form belonging to one of the domains (2).

It results from this definition:

*I. Any positive quadratic form can be transformed into an equivalent reduced form, with the help of a substitution which presents a product of substitutions belonging to a series of substitutions*

$$S_1, S_2, \dots, S_m$$

*which depend only on the choice of the system (2).*

*II. Two reduced forms can be equivalent only provided that the corresponding substitution belonged to a series of substitutions the number of which is finite.*

The weak point of the new method of reduction of positive quadratic forms, demonstrated in this Mémoire, consists in that the number of substitutions which transform into itself the domains of the set $(R)$ or their faces is, in general, very large.

The application of the general theory demonstrated in this Mémoire to the numerical examples will be particularly facilitated if one knew how to solve the following problem:

*Being given a group $G$ of substitutions which transform into itself a domain $R$, one would like to partition this domain into equivalent parts the number of which will be equal to the number of substitutions of the group $G$ and on condition that the number of faces in $\frac{n(n+1)}{2} - 1$ dimensions of domains obtained be the smallest possible.*

I show in this Mémoire the solution of the problem introduced in two cases: $n = 2$ and $n = 3$.

From the number of variable $n \geq 4$, I do not know any practical solution of the problem posed.

## First Part

### General theory of perfect positive quadratic forms and domains which correspond to them.

*Definition of perfect quadratic forms.*

1

Let

$$\phi(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j \tag{1}$$

be any positive quadratic form. By indicating with

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}) \tag{2}$$

the various representations of the minimum $M$ of the form $\sum a_{ij} x_i x_j$, one will have the equalities

$$\sum a_{ij} l_{ik} l_{jk} = M, \quad (k = 1, 2, \ldots, s) \tag{3}$$

One will not consider in the following the two systems

$$(l_{1k}, l_{2k}, \ldots, l_{nk}) \quad \text{and} \quad (-l_{1k}, -l_{2k}, \ldots, -l_{nk}), \quad (k = 1, 2, \ldots, s)$$

as different and one will arbitrarily choose one of these systems.

On the ground of the supposition made, one will have the inequality

$$\sum a_{ij} x_i x_j > M$$

provided that a system $(x_1, x_2, \ldots, x_n)$ of integral values of variables $x_1, x_2, \ldots, x_n$ did not belong to the series (2), excluding the system $x_1 = 0, x_2 = 0, \ldots, x_n = 0$.

By considering the equalities (3) as the equations which serve to determine $\frac{n(n+1)}{2}$ coefficients of the quadratic form $\sum a_{ij} x_i x_j$, one will have only two cases to examine:

1.) there exist a finite number of solutions of equations (3),

2.) the equations (3) admit only a single system of solutions.

2

Let us examine the first case,

Let us suppose that there exists an infinite number of solutions of equations (3).

One will find in this case an infinite number of values of parameters

$$p_{ij} = p_{ji}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

verifying the equations

$$\sum p_{ij} l_{ik} l_{jk} = 0, \quad (k = 1, 2, \ldots, s) \tag{4}$$

the values $p_{ij} = 0$, $i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$ being excluded.

By indicating

$$\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij} x_i x_j,$$

let us consider the set of positive quadratic forms determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n), \tag{5}$$

the parameter $\rho$ being arbitrary.

For a quadratic form determined by the equality (5) to be positive, it is necessary and sufficient that the corresponding value of the parameter $\rho$ be continuous in a certain interval

$$-R' < \rho < R.$$

It can turn out that $R = +\infty$, in this case the lower limit $-R'$ will be finite. By replacing in the equality (5) the form $\Psi(x_1, x_2, \ldots, x_n)$ by the form $-\Psi(x_1, x_2, \ldots, x_n)$, that which is permitted by virtue of (4), one will have the interval

$$-R < \rho < R',$$

therefore one can suppose that the upper limit $R$ is finite.

The corresponding quadratic form, determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n),$$

will not be positive, but it will not have negative values either; one concludes that least for a system $(\xi_1, \xi_2, \ldots, \xi_n)$ of real values of variables $x_1, x_2, \ldots, x_n$ the form $f(x_1, x_2, \ldots, x_n)$ attains in its value the smallest which is zero, and it follows that the system $(\xi_1, \xi_2, \ldots, \xi_n)$ verifies the equation

$$\frac{\partial f}{\partial \xi_i} = \frac{\partial \varphi}{\partial \xi_i} + R\frac{\partial \Psi}{\partial \xi_i} = 0. \quad (i = 1, 2, \ldots, n)$$

By eliminating from these equations $\xi_1, \xi_2, \ldots, \xi_n$ one obtains the equation

$$D(R) = \begin{vmatrix} a_{11} + RP_{11}, & a_{12} + RP_{12}, & \ldots, & a_{1n} + RP_{1n} \\ a_{21} + RP_{21}, & a_{22} + RP_{22}, & \ldots, & a_{2n} + RP_{2n} \\ & \cdots & & \\ a_{n1} + RP_{n1}, & a_{n2} + RP_{n2}, & \ldots, & a_{nn} + RP_{nn} \end{vmatrix} = 0.$$

The smallest positive root of this equation presents the value of $R$ searched for.

3

Let us examine the set $(f)$ of positive quadratic forms determined by the equality (5) with condition

$$0 < \rho < R. \tag{6}$$

*Theorem. To the set $(f)$ belongs a quadratic form $\varphi_1(x_1, x_2, \ldots, x_n)$ which is well determined by the following conditions:*

*1. all the representations of the minimum $M$ of the form $\varphi(x_1, x_2, \ldots, x_n)$ are also representations of the minimum $M$ of the form $\varphi_1(x_1, x_2, \ldots, x_n)$,*

*2. the form $\varphi_1(x_1, x_2, \ldots, x_n)$ moreover possesses at least another representation of the minimum $M$.*

Let us indicate by $M(\rho)$ the minimum and by $D(\rho)$ the determinant of the quadratic form $f(x_1, x_2, \ldots, x_n)$ defined by the equality (5) with condition (6).

By virtue fo the theorem by Hermite, one will have the inequality

$$M(\rho) \leq \mu(n) \sqrt[n]{D(\rho)}. \tag{7}$$

We have demonstrated that $D(R) = 0$, it results in that a value of the parameter $\rho$ can be chosen in the interval (6) such that the inequality

$$\mu(n) \sqrt[n]{D(\rho)} < M$$

holds. One will have, because of (7),

$$M(\rho) < M. \tag{8}$$

Let us indicate by $(l_1, l_2, \ldots, l_n)$ a representation of the minimum $M(\rho)$ of the form $f(x_1, x_2, \ldots, x_n)$ verifying the inequality (8).

One will have

$$\varphi(l_1, l_2, \ldots, l_n) + \rho\Psi(l_1, l_2, \ldots, l_n) < M, \tag{9}$$

and as a result

$$\varphi(l_1, l_2, \ldots, l_n) > M \text{ and } \Psi(l_1, l_2, \ldots, l_n) < 0. \tag{10}$$

This posed, let us find the smallest value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)} \tag{11}$$

determined with condition

$$\Psi(x_1, x_2, \ldots, x_n) \leq 0. \tag{12}$$

To that effect, let us examine the inequality

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)} \leq \frac{\varphi(l_1, l_2, \ldots, l_n) - M}{-\Psi(l_1, l_2, \ldots, l_n)}.$$

By virtue of (9), (10) and (12), one will have

$$\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) < M.$$

The quadratic form $\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n)$ being positive, there exists only a limited number of integral values of $x_1, x_2, \ldots, x_n$ verifying this inequality. Among these systems are found all the systems which give back to the function (11) the smallest value determined with condition (12).

Let us indicate by

$$(l_1', l_2', \ldots, l_n'), (l_1'', l_2'', \ldots, l_n''), \ldots, \left(l_1^{(r)}, l_2^{(r)}, \ldots, l_n^{(r)}\right)$$

all the representations of the positive minimum $\rho_1$ of the function (11).

By declaring

$$\varphi_1(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_1\Psi(x_1, x_2, \ldots, x_n),$$

one obtains the positive quadratic form $\varphi_1(x_1, x_2, \ldots, x_n)$ the minimum $M$ of which is represented by the systems (2) and (13), this is that which one will demonstrate without trouble.

4

With the help of the procedure previously shown, one will determine a series of positive quadratic forms

$$\varphi, \varphi_1, \varphi_2, \ldots \tag{14}$$

which enjoy the following property: by indicating with $s_k$ the number of representations of the minimum of the form $\varphi_k (k = 1, 2, \ldots)$, one will have the inequalities

$$s < s_1 < s_2 < \cdots \tag{15}$$

A similar series of positive quadratic forms of $n$ variables can not be extended indefinitely, this is that which we will demonstrate with the help of the following lemma.

*Lemma. The number of various representations of the minimum of a positive quadratic form in n variables does not exceed* $2^n - 1$.

Let us indicate by $(l_1, l_2, \dots, l_n)$ and $(l'_1, l'_2, \dots, l'_n)$ any two representations of the minimum $M$ of the positive quadratic form $\sum a_{ij} x_i x_j$.

Let us suppose that by declaring

$$l'_i = l_i + 2t_i, \quad (i = 1, 2, \dots, n) \tag{16}$$

the number $t_1, t_2, \dots, t_n$ would be integral.

As

$$\sum a_{ij} l'_i l'_j = M \quad \text{and} \quad \sum a_{ij} l_i l_j = M,$$

by virtue of (16), it becomes

$$\sum a_{ij} l_i t_j + \sum a_{ij} t_i t_j = 0$$

One will present this equality under the form

$$\sum a_{ij} (l_i + t_i)(l_j + t_j) + \sum a_{ij} t_i t_j = \sum a_{ij} l_i l_j. \tag{17}$$

By noticing that

$$\sum a_{ij} t_i t_j \geq \sum a_{ij} l_i l_j,$$

one finds, by virtue of (17),

$$\sum a_{ij} (l_i + t_i)(l_j + t_j) \leq 0,$$

therefore it is necessary that

$$\sum a_{ij} (l_i + t_i)(l_j + t_j) = 0,$$

and consequently

$$l_i + t_i = 0. \quad (i = 1, 2, \dots, n)$$

Because of (16), one obtains

$$l'_i = -l_i. \quad (i = 1, 2, \dots, n)$$

This posed, let us divide the set $(X)$ of all the systems $(x_1, x_2, \dots, x_n)$ of integral values of $x_1, x_2, \dots, x_n$ into $2^n$ classes, with regard to the modulo 2.

We have demonstrated that two different representations of the minimum $M$ of the form $\sum a_{ij} x_i x_j$ will not belong to the same class;

neither will any representation of the minimum $M$ belong to the class made up of systems $(x_1, x_2, \ldots, x_n)$ satisfying the conditon

$$x_i \equiv 0(\mathrm{mod}2), \quad (i = 1, 2, \ldots, n)$$

therefore the number of various representations of the minimum of a positive quadratic form can not be greater than $2^n - 1$.

5

We have demonstrated that the series (14) of positive quadratic forms satisfying the condition (15) can not be extended indefinitely, therefore the series (14) will be terminated by a form $\varphi_k$ which enjoys the following property: the form $\varphi_k$ is determined by the representations of its minimum.

*Definition. One will call perfect any positive quadratic form which is determined by the representations of its minimum.*

Let us suppose that the form (1) be perfect, one will have in this case only a single system of solutions of equations (3).

On the ground of the supposition made, the equations

$$\sum p_{ij} l_{ik} l_{jk} = 0, \quad (k = 1, 2, \ldots, s)$$

admit only a single system of solutions

$$p_{ij} = p_{ji} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By effecting the solution of equations (3), one obtains the equalities

$$a_{ij} = \alpha_{ij} M, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where the coefficients $\alpha_{ij}$ are rational.

It results in that the perfect form $\frac{\varphi}{M}$ is of rational coefficients. In the following one will not consider as different the perfect forms of proportional coefficients.

*Fundamental properties of perfect quadratic forms.*

6

Let

$$\varphi(x_1, x_2, \ldots, n) = \sum a_{ij} x_i x_j$$

be a perfect quadratic form. Let us suppose that all the different representations of the minimum of the perfect form $\varphi$ make up the series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \qquad (1)$$

By choosing any $n$ systems in this series, let us examine the determinant

$$\begin{vmatrix} l_{11}, & l_{12}, & \dots, l_{1n} \\ l_{21}, & l_{22}, & \dots, l_{2n} \\ & \dots & \\ l_{n1}, & l_{n2}, & \dots, l_{nn} \end{vmatrix} = \pm\omega. \tag{2}$$

All the determinants that one can form this way can not cancel each other out. By supposing the contrary, one will have $s$ equations of the form

$$l_{ik} = \sum_{r=1}^{n-1} l_{ir} u_r^{(k)}, \quad (i = 1, 2, \dots, n; k = 1, 2, \dots, s) \tag{3}$$

One will choose a system of $\frac{n(n+1)}{2}$ parameters $p_{ij} = p_{ji}$ verifying $\frac{n(n+1)}{2}$ equations

$$\sum p_{ij} l_{ir} l_{jt} = 0, \quad (r = 1, 2, \dots, n-1; t = 1, 2, \dots, n-1)$$

and by virtue of (3), one will have

$$\sum p_{ij} l_{ik} l_{jk} = 0, \quad (k = 1, 2, \dots, s)$$

which is impossible.

The numerical value $\omega$ of the determinant (2) can not exceed a fixed limit. To demonstrate this, let us effect a transformation of the perfect form $\varphi$ with the help of a substitution

$$x_i = \sum_{r=1}^{n} l_{ir} x'_r; \quad (i = 1, 2, \dots, n) \tag{4}$$

one will obtain a form

$$\varphi'(x'_1, x'_2, \dots, x'_n) = \sum a'_{ij} x'_i x'_j,$$

where

$$a'_{ii} = M. \ (i = 1, 2, \dots, n) \tag{5}$$

By indicating with $D'$ the determinant of the form $\varphi'$, one will have the inequality

$$a'_{11} a'_{22} \cdots a'_{nn} \geq D',$$

by virtue of the known property of positive quadratic forms.

Considering (5), one obtains

$$M^n \geq D'. \tag{6}$$

By indicating with $D$ the determinant of the form $\varphi$, one will have, because of (2) and (4),

$$D' = D\omega^2,$$

therefore the inequality (6) reduces to the one here:

$$D\omega^2 \leq M^n.$$

By virtue of the theorem by Hermite, one has the inequality

$$M \leq \mu(n)\sqrt[n]{D};$$

it follows that

$$\omega \leq [\mu(n)]^{\frac{n}{2}}.\,^{\dagger} \tag{7}$$

## 7

Any perfect form will obviously be transformed into a form, also perfect, with the help of all linear substitution of integral coefficients and of determinant ±1.

One concludes this that there exists a finite a finite number of equivalent perfect forms.

The set $(\varphi)$ of all the perfect forms in $n$ variables can be divided into different classes provided that each class be made up of all the equivalent perfect forms.

*Theorem. The number of different classes of perfect forms in n variables is finite.*

Let us indicate by

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n \quad (k = 1, 2, \ldots, s)$$

$s$ linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_s \tag{8}$$

which correspond to the systems (1) of representations of the minimum of the form $\Psi$.

One establishes this way a uniform correspondence between a perfect form $\varphi$ and the system (8) of linear forms.

Let us suppose that one had transformed the perfected form $\varphi$ with the help of a substitution $S$ by integral coefficients and with determinant ±1, one will obtain an equivalent perfect form $\varphi'$. Let us indicate by

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{9}$$

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See the Mémoire of Mr.'s Korkine and Zolotareff *sur les formes quadratiques positives.* (Mathematische Annalen V. XI, p. 256) the corresponding system of linear forms.

One will easily demonstrate that the substitution $T$, adjointed to the substitution $S$ §, will transform the system (8) into a system (9).

One concludes that a certain reduction of perfect forms can be effected with the help of the reduction of corresponding systems of linear forms.

The reduction of the system (8) comes down, by virtue of (7), to the reduction of any $n$ linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_n \tag{10}$$

belonging to the system (8) and with determinant $\pm\omega$ which does not cancel each other out.

One will determine with the help of the known method a substitution $T$ which will transform the linear forms (10) of integral coefficients into linear forms,

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_n \tag{11}$$

satisfying to the following conditions

$$\begin{cases} \lambda'_k = p_{k,k}x' + p_{k+1,k}x'_{k+1} + \ldots + p_{n,k}x'_n, \quad (k = 1, 2, \ldots, n) \\ p_{11}p_{22}\cdots p_{nn} = \omega \text{ and } p_{kk} > 0, \quad (k = 1, 2, \ldots, n) \\ 0 \leq p_{k+i,k} < p_{kk}. \quad (i = 1, 2, \ldots, n-k; k = 1, 2, \ldots, n) \end{cases}$$

The coefficients of forms (11) being integral, it results in that these coefficients do not exceed fixed limits.

The substitution $T$ will transform the system (8) into a system

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{12}$$

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§ The substitution $S$ being defined by the equalities

$$x_i = \sum_{k=1}^{n} \alpha_{ik}x'_k, \quad (i = 1, 2, \ldots, n)$$

one calls "substitution adjointed to $S$" the substitution $T$ which is determined by the equalities

$$\sum_{k=1}^{n} \alpha_{ik}x_k = x'_i. \quad (i = 1, 2, \ldots, n)$$

of linear forms. By examining successively the determinants of forms

$$(\lambda'_k, \lambda'_2, \ldots, \lambda'_n), (\lambda'_1, \lambda'_k, \ldots, \lambda'_n), \ldots, (\lambda'_1, \lambda'_2, \ldots, \lambda'_k), \; (k = n+1, n+2, \ldots, s)$$

one will demonstrate that the numerical values of coefficients of all the linear forms (12) do not exceed fixed limits.

The number of similar systems of linear forms in integral coefficients being limited, it results in that the number of different classes of perfect forms is also limited.

### *On the domains determined with the help of linear inequalities*

8

We have seen in Number 7 that the study of perfect forms can be brought back to the study of certain systems of linear forms.

One will acquire a new basis to these studies by making correspond to each perfect quadratic form in $n$ variables a domain in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

One will address first the general problem by studying the properties of domains determined with the help of linear inequalities. ‡

Let us consider a system of linear inequalities

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

in any real coefficients.

One will call point $(x)$ any system $(x_1, x_2, \ldots, x_m)$ of real values of variables $x_1, x_2, \ldots, x_m$ and one will indicate

$$y_k(x) = p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m. \quad (k = 1, 2, \ldots, \sigma)$$

One will call "domain" the set $R$ of points verifying the inequalities

$$y_k(x) \geq 0. \; (k = 1, 2, \ldots, \sigma) \tag{1}$$

Let us suppose that to the domain $R$ belonged to points verifying the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \ldots, \sigma)$$

one will call such points interior to the domain $R$, and the domain $R$ will be said to be of $m$ dimensions.

It can be the case that the domain $R$ does not possess interior points. One will demonstrate in this case all the points belonging to the domain $R$ verify at least one equation

$$y_k(x) = 0,$$

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‡ See: *Minkowski.* Geometrie der Zahlen [Geometry of the numbers], No. 19, p. 39.

the indice $h$ being a value $1, 2, \ldots, \sigma$.

It is important to have a criteria with the help of which one could recognise whether a domain determined by the help of inequalities (1) will be in $m$ dimensions or not.

*Fundamental principle. For a domain determined with the help of inequalities (1) to be of $m$ dimensions, it is necessary and sufficient that the equation*

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \qquad (\Xi)$$

*did not reduce into an identity so long as the parameters $\rho_2, \rho_2, \ldots, \rho_\sigma$ are positive or zero, the values $\rho_1 = 0, \rho_2 = 0, \ldots, \rho_\sigma = 0$ being excluded.*

The principle introduced, considered from a certain point of view, is evident, but one arrive at the rigorous demonstration of this principle only with the help of the in depth study of domains determined with the help of linear inequalities.

For more simplicity, one will examine in that which follows only domains satisfying the following conditions: the equations

$$y_k(x) = 0 \;\; (k = 1, 2, \ldots, \sigma) \qquad (2)$$

can not be verified by any point, the point $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ being excluded.

It is easy to demonstrate that the general case will always come down to the case examined.

9

*Definition. One will call edge of the domain $R$ determined with the help of inequalities (1) the set of points belonging to the domain $R$ and verifying the equations*

$$y_k(x) = 0, \quad (k = 1, 2, \ldots, r \text{ where } r < \sigma)$$

*provided that these equations defined the values of $x_1, x_2, \ldots, x_m$ to an immediate common factor.*

By indicating with $(\xi_1, \xi_2, \ldots, \xi_m)$ a point of the edge considered, one will determine all the points of the edge with the help of equalities

$$x_i = \rho \xi_i, \quad (i = 1, 2, \ldots, n)$$

$\rho$ being an arbitrary positive parameter.

This results in that each edge of the domain $R$ is well determined by any point belonging to it.

Let us suppose that the domain $R$ possesses $s$ edges characterised by the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \dots, \xi_{mk}). \quad (k = 1, 2, \dots, s)$$

By declaring

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik}, \quad (i = 1, 2, \dots, m) \tag{3}$$

where

$$\rho_k \geq 0, \quad (k = 1, 2, \dots, s) \tag{4}$$

one obtains a point $(x)$ belonging to the domain $R$, the positive or zero parameters $\rho_1, \rho_2, \dots, \rho_s$ being arbitrary.

10

*Fundamental theorem. Let us suppose that the inequalities (1) which define the domain $R$ satisfy the condition ($\Xi$).*

*The domain $R$ will be of $m$ dimensions and each point belonging to it will be determined by the equalities (3) with condition (4).*

The theorem introduced is well known in the case $m = 2$ and $m = 3$.

We will demonstrate that by supposing that the theorem be true in the case of $m - 1$ variables, the theorem will again be true in the case of $m$ variables.

Let us examine first the various inequalities of the system (1). It can be the case that many among them could be put under the form

$$y_h(x) = \sum_{k=1}^{s} \rho_k^{(h)} y_k(x) \quad \text{where} \quad \rho_k^{(h)} \geq 0. \quad (k = 1, 2, \dots, s; \rho_h^{(h)} = 0)$$

One will call such inequalities dependent and one will exclude them from the system (1).

Let us suppose that the system (1) contained only independent inequalities.

Their number $\rho$, on the ground of the supposition (2) made, will not be less than $m$.

This posed, let us examine a set $P_h$ of points belonging to the domain $R$ and verifying an equation

$$y_h(x) = 0, \tag{5}$$

the indice $h$ having a value $1, 2, \dots, \sigma$.

One will call "face of the domain $R$" the domain $P_h$.

On the ground of the supposition made, the face $P_h$ will be in $m-1$ dimensions.

To demonstrate this, let us make correspond to any point $(x)$ verifying the equation (5) a point $(u)$ in $m-1$ coordinates $(u_1, u_2, \dots, u_{m-1})$ by declaring

$$x_i = \sum_{j=1}^{m-1} \alpha_{ij} u_j. \quad (i = 1, 2, \dots, m)$$

The system of inequalities (1) will be transformed into a system

$$\eta_k(u) \geq 0 \quad (k = 1, 2, \dots, \sigma; k \neq h) \tag{7}$$

of inequalities in $m-1$ variables $u_1, u_2, \dots, u_{m-1}$.

Let us suppose that one knew how to reduce the equation

$$\sum_{k=1}^{\sigma} \rho_k \eta_k(u) = 0 \text{ where } \rho_h = 0 \text{ and } \rho_k \geq 0 \ (k = 1, 2, \dots, \sigma) \tag{8}$$

into an identity. By virtue of (6), one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = \rho y_h(x) \quad \text{where} \quad \rho_h = 0.$$

One can not suppose that $\rho > 0$, since otherwise the inequality

$$y_h(x) \geq 0$$

would be dependent and on the ground of the supposition made would not belong to the system (1).

By supposing that $\rho \leq 0$, one will admit $\rho_h = -\rho$ and one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \dots, \sigma)$$

which is contrary to the hypothesis.

We have supposed that the theorem introduced be true in the case of $m-1$ variables. As the equation (8) can not be reduced into an identity, one concludes that the system of inequalities (7) defines a domain $\mathcal{B}_h$ in $m-1$ dimensions. Moreover, by indicating with

$$(u_{11}, u_{21}, \dots, u_{m-1,1}), (u_{12}, u_{22}, \dots, u_{m-1,2}), \dots, (u_{1t}, u_{2t}, \dots, u_{m-1,t}) \tag{9}$$

the points which characterised $t$ edges of the domain $\mathcal{B}_h$, one will determine any point $(u)$ of this domain by the equalities

$$u_i = \sum_{k=1}^{t} \rho_k u_{ik} \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m-1) \tag{10}$$

One will make correspond to the points (9) the points

$$(\xi_r) = (\xi_{1r}, \xi_{2r}, \ldots, \xi_{mr}), \quad (r = 1, 2, \ldots, t) \tag{11}$$

by determining them with the help of equalities (6) and (9).

The points obtained (11) characterise $t$ edges of the domain $R$ belonging to the face $P_h$. Any point $(x)$ belonging to the face $P_h$ will be determined, on the grounds of (6) and (10), by the equalities

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \tag{12}$$

Let us notice that all the points (11) verify the equation

$$y_h(x) = 0 \tag{13}$$

and satisfy the conditions

$$y_k(x) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One obtains thus the equalities

$$y_h(\xi_r) \geq 0 \quad (r = 1, 2, \ldots, t; k = 1, 2, \ldots, \sigma) \tag{14}$$

The face $P_h$ being in $m-1$ dimensions, the equalities (14) would define the coefficients of the equation (13) to a close by common factor.

11

Let us suppose that one had determined this way all the faces

$$P_1, P_2, \ldots, P_\sigma \tag{15}$$

in $m-1$ dimensions of the domain $R$.

Let us suppose that the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}) \ (k = 1, 2, \ldots, s) \tag{16}$$

characterise the various edges of the domain $R$ belonging to the various faces (15).

By indicating

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m) \tag{17}$$

one obtains a set of points which all belong to the domain $R$.

I say that any point $(x)$ belonging to the domain $R$ can be determined with the help of equalities (17).

One can suppose that the point $(x)$ does not belong to any one of the faces (15), since any point belonging to them can be determined with the help of equalities (12).

By supposing that one had the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \ldots, \sigma)$$

let us arbitrarily choose a point $(\xi_r)$ among those of the series (16) and let us admit

$$x'_i = x_i - \rho\xi_{ir} \text{ where } \rho > 0. \ (i = 1, 2, \ldots, m) \tag{18}$$

So long as the parameter $\rho$ is sufficiently small, one will also have

$$y_k(x') > 0. \quad (k = 1, 2, \ldots, \sigma)$$

By making the parameter increase in a continuous manner, one will determine with the help of equalities (18) a point $(x')$ verifying an equation

$$y_h(x') = 0$$

and satisfying the condition

$$y_k(x') \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

The point obtained $(x')$ belongs to the face $P_h$, therefore one can declare

$$x'_i = \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho'_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

By virtue of (18), it becomes

$$x_i = \rho\xi_{ir} + \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho > 0, \ \rho'_k \geq 0. \ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

It remains to demonstrate that the domain $F$ is in $m$ dimensions.

Let us notice that all the points determined by the equalities (17) with condition

$$\rho_k > 0 \quad (k = 1, 2, \ldots, s)$$

are interior to the domain $R$.

In effect, all the points (16) verify the inequalities

$$y_h(\xi_k) \geq 0. \ (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma) \tag{19}$$

By multiplying these inequalities by $\rho_k$, let us make the sum of inequalities obtained; one will have, because of (17),

$$y_h(x) = \sum_{k=1}^{s} \rho_k y_h(\xi_k) \geq 0. \quad (h = 1, 2, \ldots, \sigma)$$

By virtue of (19), one will have the inequality

$$y_h(x) > 0, \ \ (h = 1, 2, \ldots, s) \tag{20}$$

so long as the numbers $y_h(\xi_1), y_h(\xi_2), \ldots, y_h(\xi_s)$ do not cancel each other out.

One can not suppose that the equalities

$$y_h(\xi_k) = 0 \quad (k = 1, 2, \ldots, s)$$

holds, because otherwise all the equations

$$y_1(x) = 0, y_2(x) = 0, \ldots, y_\sigma(x) = 0$$

would be of proportional coefficients, which is contrary to the hypothesis; therefore one will have the inequalities (20), and it follows that the domain $R$ is of $m$ dimensions.

We have demonstrated that the condition $(\Xi)$ is sufficient for the domain $R$ to be of $m$ dimensions. It is easy to demonstrate that this condition is necessary.

12

We have defined in Number 10 the faces in $m-1$ dimensions of the domain $R$. This definition can be generalised.

*Definition. One will call face in $\mu$ dimensions of the domain $R$ $(\mu = 1, 2, \ldots, m-1)$ a domain $P(\mu)$ formed from points belonging to the domain $R$ and verifying a system of equations*

$$y_k(x) = 0, \ \ (k = 1, 2, \ldots, \tau) \tag{21}$$

*provided that these equations define a domain in $\mu$ dimensions composed of points which, all, do not verify any other equation $y_{\tau+1}(x) = 0, \ldots, y_\sigma(x) = 0$.*

Let us choose among the points (16) all those which verify the equations (21).

By indicating with

$$\xi_k = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}), \quad (k = 1, 2, \ldots, t)$$

one will declare

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \quad (22)$$

It is easy to demonstrate that any point $(x)$ belonging to the face $P(\mu)$ can be determined with the help of equalities (22).

*Corollary. Each face of the domain $R$ is a set of points determined by the equalities (22) provided that any point belonging to it could not be determined by the equalities*

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k \geq 0, \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m)$$

*unless all the parameters $\rho_{t+1}, \rho_{t+2}, \ldots, \rho_s$ do not cancel each other.*

13

Any point belonging to the domain $R$ either is interior to the domain $R$ or is interior to a face of that domain.

Let us suppose that the point $(x)$ be interior to a face $P(\mu)$ of the domain $R$ which is formed from all the points determined by the equalities (22).

I argue that one can always determine the point $(x)$ by the equalities (22) provided that

$$\rho_k > 0. \quad (k = 1, 2, \ldots, t)$$

To demonstrate this, let us indicate

$$\rho_i = \sum_{k=1}^{t} \xi_{ik}. \quad (i = 1, 2, \ldots, m)$$

The point $(\alpha)$ is interior to the face $P(\mu)$.

By admitting

$$x'_i = x_i - \rho\alpha_i \text{ where; } \rho > 0, \ (i = 1, 2, \ldots, m) \quad (23)$$

one obtains a point $(x'_i)$ which will be interior to the face $P(\mu)$ so long as the parameter $\rho$ will be sufficiently small; it follows that

$$x'_i = \sum_{k=1}^{t} \rho'_k \xi_{ik} \quad \text{where} \quad \rho'_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

By virtue of (23), one obtains

$$x_i = \sum_{k=1}^{t} (\rho + \rho'_k) \xi_{ik}, \quad (i = 1, 2, \ldots, m)$$

and by making

$$\rho + \rho'_k = \rho_k, \quad (k = 1, 2, \ldots, t)$$

one will have

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k > 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

Let us notice that by making $\mu = m$ and $t = s$, one will indicate with the symbol $P(m)$ the domain $R$; one concludes that any point $(x)$ which is interior to the domain $R$ can be determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k > 0. \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m)$$

*On the correlative domains.*

14

*Definition. Let us suppose that a domain $R$ be determined with the help of inequalities*

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call correlative to the domain $R$ the domain $\mathcal{R}$ which is formed from all the points $(x)$ determined by the equalities

$$x_i = \sum_{k=1}^{\sigma} \rho_k p_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, m) \quad (1)$$

I say that the domain $\mathcal{R}$ will be in $m$ dimensions, if the domain $R$ does not possess points verifying the equations

$$p_{1k}x_1 + p_{1k}x_2 + p_{mk}x_m = 0, \quad (k = 1, 2, \ldots, \sigma)$$

the point $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ being excluded.

In effect, if all the points of the domain $\mathcal{R}$ verified the same equation

$$\xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_m x_m = 0,$$

one would have the equalities

$$\xi_1 p_{1k} + \xi_2 p_{2k} + \ldots + \xi_m p_{mk} = 0, \quad (k = 1, 2, \ldots, \sigma)$$

by virtue of (1), which is contrary to the hypothesis.

*Theorem. By supposing that the domain $R$ be formed from all the points $(x)$ determined by the equalities*

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m) \qquad (2)$$

*one will define the correlative domain $\mathcal{R}$ with the help of inequalities.*

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \ldots + \xi_{mk}x_m \geq 0. \ (k = 1, 2, \ldots, s) \qquad (3)$$

Let us indicate by $\mathcal{R}'$ thedomain determined with the help of inequalities (3).

On the ground of the supposition made, all the points

$$(\xi_{11}, \xi_{21}, \ldots, \xi_{m1}), (\xi_{12}, \xi_{22}, \ldots, \xi_{m2}), \ldots, (\xi_{1s}, \xi_{2s}, \ldots, \xi_{ms})$$

characterise the edges of the domain $R$, and one will have the inequalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \ldots + p_{mh}\xi_{mk} \quad (h = 1, 2, \ldots, \sigma; h = 1, 2, \ldots, \sigma). \ (4)$$

We have seen in Number 10 that each face $P_h$ in $m-1$ dimensions of the domain $R$ is characterised by the points

$$(\xi_{11}, \xi_{21}, \ldots, \xi_{m1}), (\xi_{12}, \xi_{22}, \ldots, \xi_{m2}), \ldots, (\xi_{1t}, \xi_{2t}, \ldots, \xi_{mt})$$

which verify the equation

$$y_h^x = 0 \qquad (5)$$

of the face $P_h$. One obtains the equalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \ldots + p_{mh}\xi_{mk} = 0 \quad (k = 1, 2, \ldots, t)$$

which define the coefficients $p_{1h}, p_{2h}, \ldots, p_{mh}$ of the equation (5) to an immediate common factor.

One concludes, by virtue of the definition established in Number 9, that the point $(p_{1h}, p_{2h}, \ldots, p_{mh})$ characterises an edge of the domain $\mathcal{R}'$.

By attributing with the indice $h$ the values $1, 2, \ldots, \sigma$, one obtains a series

$$(p_{11}, p_{21}, \ldots, p_{m1}), (p_{12}, p_{22}, \ldots, p_{m2}), \ldots, (p_{1\sigma}, p_{2\sigma}, \ldots, p_{m\sigma})$$

of points which characterise different edges of the domain $\mathcal{R}'$.

I argue that the domain $\mathcal{R}'$ does not possess other edges. To demonstrate this, let us suppose that a point $p_1, p_2, \ldots, p_m$ characterises an edge of the domain $\mathcal{R}'$.

One will have the equalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \ldots + p_m\xi_{mh} = 0, \quad (h = 1, 2, \ldots, t) \qquad (6)$$

which define the coefficients $p_1, p_2, \ldots, p_m$ to a nearby common factor, and one will have the inequalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \ldots + p_m\xi_{mh} \geq 0. \quad (k = 1, 2, \ldots, s) \qquad (7)$$

Let $(x)$ be any point of the domain $R$. One will determine the point $(x)$ with the help of equalities (2). By multiplying the inequalities (7) with $\rho_k$ and by making the sum of inequalities obtained, one will have, because of (2),

$$p_1x_1 + p_2x_2 + \ldots + p_mx_m \geq 0.$$

One concludes that the inequalities

$$-p_1x_1 - p_2x_2 - \ldots - p_mx_m \geq 0 \text{ and } p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0,$$

define a domain which is not in $m$ dimensions.

By virtue of the fundamental theorem of Number 10, one will determine in this case positive values or zeros of parameters $\rho, \rho_1, \ldots, \rho_\sigma$ which reduce the equation

$$-\rho(p_1x_1 + p_2x_2 + \ldots + p_mx_m) + \sum_{k=1}^{\sigma} \rho_k(p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m) = 0$$

into an identity.

It follows that

$$p_i = \sum_{k=1}^{\sigma} \frac{\rho_k}{\rho} p_{ik} \text{ where } \frac{\rho_k}{\rho} \geq 0. \quad (k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, m)$$

By substituting (6), one will have

$$\sum_{k=1}^{\sigma} \frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk}) = 0. \quad (h = 1, 2, \ldots, t)$$

By virtue of (4), one finds

$$\frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk}) = 0. \ (h = 1, 2, \ldots, t; k = 1, 2, \ldots, \sigma)$$

Let us suppose that $\frac{\rho_k}{\rho} > 0$, then

$$\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk} = 0, \quad (h = 1, 2, \ldots, t)$$

therefore the coefficients $p_1, p_2, \ldots, p_m$, by virtue of (6), are proportional to the coefficients $p_{1k}, p_{2k}, \ldots, p_{mk}$; it follows that the points $(p_{1k}, p_{2k}, \ldots, p_{mk})$ and $(p_1, p_2, \ldots, p_m)$ characterise the same edge of the domain $\mathcal{R}'$.

By virtue of the fundamental theorem in Number 10, all the points of the domain $\mathcal{R}'$ will be determined by the equality (1), this results in that the domains $\mathcal{R}$ and $\mathcal{R}'$ coincide.

*Corollary. Let us suppose that a face $P(\mu)$ in $\mu$ dimensions of the domain $R$ be determined by the equations*

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m = 0, \quad (k = 1, 2, \ldots, \tau)$$

*and that any point $(x)$ belonging to the face $P(\mu)$ be determined by the equalities*

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

*The correlative domain $\mathcal{R}$ will possess a corresponding face $\mathcal{B}(m-\mu)$ in $m-\mu$ dimensions determined by the equations*

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \ldots + \xi_{mk}x_m = 0 \quad (k = 1, 2, \ldots, t)$$

*and any point $(x)$ belonging to the face $\mathcal{B}(m-\mu)$ will be determined by the equalities*

$$x_i = \sum_{k=1}^{\tau} \rho_k p_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, \tau; i = 1, 2, \ldots, m)$$

### *Definition of domains of quadratic forms corresponding to the various perfect forms*

15

Let us consider any one perfect quadratic form $\varphi$.

Let us suppose that all the representations of the minimum of the form $\varphi$ make up the series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \qquad (1)$$

By indicating

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n, \quad (k = 1, 2, \ldots, s) \qquad (2)$$

one corresponds to the series (1) a series of linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_s.$$

Let us consider a domain $R$ of quadratic forms determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{s} \rho_k \lambda_k^2$$

with condition that

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, s)$$

One will say that the domain $R$ correspond to the perfect form $\varphi$.

Let us notice that the domain $R$ is in $\frac{n(n+1)}{2}$ dimensions.

By supposing the contrary let us suppose that all the quadratic forms belonging to the domain $R$ verifies a linear equation

$$\Psi(f) = \sum p_{ij} a_{ij} = 0.$$

On the ground of the established definition, one will have the equalities

$$\Psi(\lambda_k^2) = 0 \quad (k = 1, 2, \ldots, s)$$

or, that which comes to the same thing, because of (2),

$$p_{ij} l_{ik} l_{jk} = 0 \quad (k = 1, 2, \ldots, s)$$

which is impossible, the form $\varphi$ being perfect.

On the ground of what has been said in Number 9–14, the domain $R$ possesses $s$ edges characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_s^2. \tag{3}$$

Let us suppose that one had determined all the faces

$$P_1, P_2, \ldots, P_\sigma$$

in $\frac{n(n+1)}{2} - 1$ dimensions of the domain $R$.

Each face $P_k$ can be determined by two methods:

1. All the quadratic forms belonging to the face $P_k$ verify an equation

$$\Psi_k(f) = \sum p_{ij}^{(k)} a_{ij} = 0$$

which can be determined in such a way that the inequality

$$\Psi_k(f) > 0$$

held so long as the form $f$ belonging to the domain $R$ is exterior to the face $P_k$.

2. By choosing among the quadratic forms (3) these

$$\lambda_1^2, \lambda_2^2, \dots, \lambda_t^2$$

which verify the equation (4), one will determine all the quadratic forms belonging to the face $P_k$ by the equalities

$$f(x_1, x_2, \dots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2,$$

where

$$\rho_k \geq 0. \quad (k = 1, 2, \dots, t)$$

By virtue of the theorem of Number 14, the domain $R$ can be considered as a set of points verifying the inequalities

$$\Psi_k(f) \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

### *On the extreme quadratic forms*

16

Let us indicate by $M(a_{ij})$ the minimum and by $D(a_{ij})$ the determinant of a positive quadratic form $\sum a_{ij}x_ix_j$. The positive quadratic form $\frac{1}{\sqrt[n]{D(a_{ij})}} \sum a_{ij}x_ix_j$ will be of determinant 1 and will possess the minimum

$$\frac{M(\alpha_{ij})}{\sqrt[n]{D(a_{ij})}} = \mathcal{M}(a_{ij}).$$

Let us examine the various value of the function $\mathcal{M}(a_{ij})$ which is well determined in the set $(f)$ of all the positive quadratic forms in $n$ variables.

*Definition. One will call extreme* ‡ *a positive quadratic form* $\sum a_{ij}x_ix_j$ *which enjoys the property that the corresponding value of the function* $\mathcal{M}(a_{ij})$ *is minimum.*

Let us notice that the function $\mathcal{M}(a_{ij})$ does not change its value when one replaces quadratic form $\sum a_{ij}x_ix_j$ by a form of proportional coefficients.

By attributing to the coefficients of the extreme form $\sum a_{ij}x_ix_j$ variations

$$\epsilon_{ij} = \epsilon_{ji} \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n)$$

satisfying the condition

$$|\epsilon_{ij}| < \epsilon, \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n) \qquad (1)$$

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‡ See the Mémoire of Mr.'s Korkine and Zolotareff, Sur les formes quadratiques [On the quadratic forms], Mathematische Annalen V. VI, p. 368

$\epsilon$ being an arbitrary positive parameter, let us examine the corresponding value of the function $\mathcal{M}(a_{ij})$.

On the ground of the definition established, one can determine the parameter $\epsilon$ such that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) < \mathcal{M}(a_{ij}) \tag{2}$$

held with condition (1) and so long as the coefficients $\epsilon_{ij}$ are not proportional to the coefficients

$$a_{ij}. \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n)$$

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*Theorem. For a quadratic form $\sum a_{ij}x_ix_j$ to be extreme, it is necessary and sufficient that it be perfect and that its adjointed form $\sum \frac{\partial D(a_{ij})}{\partial a_{ij}} x_ix_j$ be interior to the domain corresponding to the form $\sum a_{ij}x_ix_j$.*

Let us indicate by

$$(l_{11}, l_{21}, \dots, l_{n1}), (l_{12}, l_{22}, \dots, l_{n2}), \dots, (l_{1s}, l_{2s}, \dots, l_{ns}) \tag{3}$$

the various representations of the minimum $M(a_{ij})$ of the form $\sum a_{ij}x_ix_j$.

Let us consider a quadratic form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$, the parameter $\rho$ being arbitrary. One can determine an interval

$$-\delta < \rho < \delta \text{ where } 0 < \delta < 1 \tag{4}$$

such that all the representations of the minimum of the form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$ are found among the systems (3) so long as the variations $\epsilon_{ij}$ satisfy the condition (1).

By indicating with

$$M' = \sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk} \text{ and } M = \sum a_{ij}l_{ik}l_{jk} \tag{5}$$

the minima of forms $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$ and $\sum a_{ij})x_ix_j$ and with $D'$ and $D$ their determinants, one will have

$$\mathcal{M}(a_{ij} + \rho\epsilon_{ij}) = \frac{\sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}}, \quad \mathcal{M}(a_{ij}) = \frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}.$$

By virtue of (2), one obtains the inequality

$$\frac{\sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}} < \frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}$$

or, that which comes to the same thing,

$$\rho \sum \epsilon_{ij} l_{ik} l_{jk} < M \left( \sqrt[n]{\frac{D'}{D}} - 1 \right). \tag{6}$$

This declared, let us suppose that the form $\sum a_{ij} x_i x_j$ be not perfect.

One will determine in this case the variations $\epsilon_{ij}$ such that the equalities

$$\sum \epsilon_{ij} l_{ik} l_{jk} = 0,$$

held. By virtue of (6), one will obtain the inequality

$$D' > D.$$

By developping the determinant $D'$ into a series, one will have the inequality

$$\rho \sum \epsilon_{ij} \frac{\partial D}{\partial a_{ij}} + \frac{\rho^2}{2} \sum \epsilon_{ij} \epsilon_{kh} \frac{\partial^2 D}{\partial a_{ij} \partial a_{kh}} + \ldots > 0. \tag{7}$$

The parameter $\rho$ being arbitrary satisfying the condition (4), it is necessary tht

$$\sum \epsilon_{ij} \frac{\partial D}{\partial a_{ij}} = 0.$$

Mr.'s Korkine and Zolotareff have demonstrated ‡ that in this case one will always have the inequality

$$\sum \epsilon_{ij} \epsilon_{kh} \frac{\partial^2 D}{\partial a_{ij} \partial a_{kh}} < 0,$$

therefore the inequality (7) is impossible.

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We have demonstrated that the form $\varphi = \sum a_{ij} x_i x_j$ has to be perfect.

Let us suppose that the domain $R$ corresponding to the perfect form $\varphi$ be determined by $\sigma$ inequalities

$$\Psi_r(f) = \sum p_{ij}^{(r)} a_{ij} \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

On the grounds of these inequalities, one will have

$$\Psi_r(\lambda_k^2) = \sum p_{ij}^{(r)} l_{ik} l_{jk} \geq 0. \;\; (k = 1, 2, \ldots, s; r = 1, 2, \ldots, \sigma) \tag{8}$$

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‡ Mathematische Annalen, V. XI, p. 250

Let us declare

$\epsilon_{ij} = tp_{ij}^{(r)}$ where $t > 0$. $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$

By virtue of (6), one will have

$$\rho t \sum p_{ij}^{(r)} l_{ik} l_{jk} < M \left( \sqrt[n]{\frac{D'}{D}} - 1 \right). \tag{9}$$

Let us attribute to the parameter $\rho$ a positive value satisfying the condition (4), by virtue of (8) and (9) there will arrive

$$D' > D.$$

By developping the determinant $D'$ into a series, one obtains the inequality

$$\rho t \sum p_{ij}^{(r)} \frac{\partial D}{\partial a_{ij}} + \frac{(\rho t)^2}{2} \sum p_{ij}^{(r)} p_{ih}^{(r)} \frac{\partial^2 D}{\partial a_{ij} \partial a_{kh}} + \ldots > 0.$$

The positive parameter $\rho$ being as small as one wish, it follows that

$$\sum p_{ij}^{(r)} \frac{\partial D}{\partial a_{ij}} > 0. \quad (r = 1, 2, \ldots, \sigma)$$

It is thus demonstrated that the form $\sum \frac{\partial D}{\partial a_{ij}} x_i x_j$, adjointed to the form $\varphi$, is interior to the domain $R$.

I argue that in this case the perfect form $\varphi$ will be extreme.

By supposing the contrary, let us suppose that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) \geq \mathcal{M}(a_{ij}) \tag{10}$$

be verified by any one system of variations $\epsilon_{ij}$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ satisfying the condition (1) however small the parameter $\epsilon$ may be.

By virtue of (10), one obtains

$$\sum \epsilon_{ij} l_{ik} l_{jk} \geq M \left( \sqrt[n]{\frac{D'}{D}} - 1 \right); \quad (k = 1, 2, \ldots, s) \tag{11}$$

the inequality obtained has to hold whatever may te the value of the index $k = 1, 2, \ldots, s$.

By indicating

$$\eta_{ij} = a_{ij} \left( \sqrt[n]{\frac{D}{D'}} - 1 \right) + \epsilon_{ij} \sqrt[n]{\frac{D}{D'}}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{12}$$

let us examine the quadratic form

$$\varphi_0(x_1, x_2, \ldots, x_n) = \sum (a_{ij} + \eta_{ij}) x_i x_j. \tag{13}$$

By virtue of (12) the form $\varphi_0$ is of determinant $D$.

By choosing the parameter $\epsilon$ sufficiently small, one can suppose that

$$|\eta_{ij}| < \eta, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{14}$$

$\eta$ being a positive parameter as small as one would like.

By virtue of (5), (11) and (12), one obtains

$$\sum \eta_{ij} l_{ik} l_{jk} \geq 0. \ \ (k = 1, 2, \ldots, s) \tag{15}$$

By developping the determinant $D$ of the form (13) into series, one will find

$$D + \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} + R_2 = D. \tag{16}$$

In this equality the remainder $R_2$ verifies an inequality

$$|R_2| < \eta^2 P,$$

$P$ being a positive number not depending on the parameter $\eta$ so long as $\eta < 1$.

By virtue of (16), one obtains

$$\left| \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} \right| < \eta^2 P. \tag{17}$$

We have suppose that the quadratic form $\sum \frac{\partial D}{\partial a_{ij}} x_i x_j$, adjointed to the form $\varphi$, be interior to the domain $R$. On the ground of that which has been said in Number 13, one will determine the form $\sum \frac{\partial D}{\partial a_{ij}} \cdot x_i x_j$ with the help of the equality

$$\sum \frac{\partial D}{\partial a_{ij}} x_i x_j = \sum_{k=1}^{s} \rho_k \lambda_k^2 \tag{18}$$

where

$$\rho_k > 0. \ \ (k = 1, 2, \ldots, s) \tag{19}$$

The equality (18) can be replaced by the following ones:

$$\frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k l_{ik} l_{jk}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By multiplying these equations by $\eta_{ij}$ and by adding up the equalities obtained, one will have

$$\sum \eta_{ij}\frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k \sum \eta_{ij} l_{ik} l_{jk}. \tag{20}$$

By virtue of (15), (17) and (19), one obtains the inequalities

$$0 \leq \sum \eta_{ij} l_{ik} l_{jk} < \eta^2 \frac{P}{\rho_k}; \quad (k = 1, 2, \ldots, s)$$

therefore one can admit

$$\sum \eta_{ij} l_{ik} l_{jk} = \tau_k \eta^2, \ \ (k = 1, 2, \ldots, s) \tag{21}$$

and the positive numbers or zeros $\tau_k$ $(k = 1, 2, \ldots, s)$ will not exceed fixed limits which do not depend on the parameter $\eta$.

After the definition of perfect forms, the equations (21) admit only a single system of solutions. By effecting this solution of equations (21), one obtains

$$\eta_{ij} = \tau_{ij}\eta^2 \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where

$$|\tau_{ij}| < T, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

$T$ being a positive number which does not depend on the parameter $\eta$; therefore one will have the inequalities

$$|\eta_{ij}| < \eta^2 T. \ \ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{22}$$

This stated, let us take any one positive fraction $\vartheta$ and declare

$$\eta = \frac{\vartheta}{T}.$$

By virtue of (14), one will have

$$|\eta_{ij}| < \frac{\vartheta}{T}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

and because of (22), it will become

$$|\eta_{ij}| < \frac{\vartheta^2}{T}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By admitting

$$\eta_{ij} < \frac{\vartheta^2}{T},$$

one will have, because of (22),

$$|\eta_{ij}| < \frac{\vartheta^4}{T}, \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n)$$

and so on.

One will obtain in this manner the inequalities

$$|\eta_{ij}| < \frac{\vartheta^{2^k}}{T} \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n; k = 0, 1, 2, \dots)$$

it follows that

$$\eta_{ij} = 0. \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n)$$

By virtue of (12), one obtains

$$\epsilon_{ij} = a_{ij}\left(\sqrt[n]{\frac{D'}{D}} - 1\right); \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n)$$

therefore the coefficients $\epsilon_{ij}$ are proportional to the coefficients $a_{ij}$ $(i = 1, 2, \dots, n; j = 1, 2, \dots, n)$, which is contrary to the hypothesis.

*Properties of the set of domains corresponding to the various perfect forms in n variables.*

19

Any perfect form $\varphi$ will be transformed by an equivalent perfect form $\varphi'$ with the help of any substitution $S$ of integral coefficients and of determinant ±1.

Let us indicate by $R$ and $R'$ the domains corresponding to the perfect forms $\varphi$ and $\varphi'$ and by $T$ the substitution adjointed to the substitution $S$.

One will easily demonstrate that the domain $R$ will be transformed into an equivalent domain $R'$ with the help of the substitution $T$.

One concludes that there exists a finite number of domains equivalent to the domain $R$.

Let us indicate by $(R)$ the set of all the domains corresponding to the various perfect forms in $n$ variables.

The set $(R)$ can be divided into classes of equivalent domains.

On the ground of that which has been previously said, the number of different classes of the set $(R)$ is equal to the number of classes of perfect forms in $n$ variables.

20

*Theorem. Let us suppose that a quadratic form $f$ be interior to a face $P(\mu)$ in $\mu$ dimensions of the domain $R$ ($\mu = 1, 2, \ldots, \frac{n(n+1)}{2}$).*

*The form $f$ will belong only to the domains of the set $(R)$ which are contiguous through the face $P(\mu)$.*

Let us suppose that the domain $R$ be characterised by the quadratic form

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_s^2 \tag{1}$$

and that the face $P(\mu)$ in $\mu$ dimensions of the domain $R$ be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2. \tag{2}$$

In the case $\mu = \frac{n(n+1)}{2}$, one will admit $t = s$, and the symbol $P\left(\frac{n(n+1)}{2}\right)$ will indicate the domain $R$.

The quadratic form $f$ being interior to the face $P(\mu)$, one will have the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2 \text{ where } \rho_k > 0. \ (k = 1, 2, \ldots, t) \tag{3}$$

Let us suppose that the same form $f$ belonged to another domain $R'$ of the set $(R)$.

Let us suppose that the domain $R'$ be characterised by the quadratic forms

$${\lambda'_1}^2, {\lambda'_2}^2, \ldots, {\lambda'_\sigma}^2 \tag{4}$$

and that the form $f$ be interior to the face $P'(\nu)$ of the domain $R'$ characterised by the quadratic forms

$${\lambda'_1}^2, {\lambda'_2}^2, \ldots, {\lambda'_\tau}^2. \tag{5}$$

One will have, on the ground of the supposition made,

$$f(x_1, x_2, \ldots, x_n) = \sum_{h=1}^{\tau} \rho'_h {\lambda'_h}^2 \text{ where } \rho'_h > 0. \ (h = 1, 2, \ldots, \tau) \tag{6}$$

This declared, let us indicate by $\varphi$ and $\varphi'$ the perfect forms corresponding to the domains $R$ and $R'$ and suppose, for more simplicity, that the minimum of forms $\varphi$ and $\varphi'$ be $M$.

By indicating with the symbol $(f, f')$ the result

$$(f, f') = \sum a_{ij} a'_{ij},$$

from two quadratic forms

$$f(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j \text{ and } f'(x_1, x_2, \ldots, x_n) = \sum a'_{ij} x_i x_j,$$

let us examine two results $(f, \varphi)$ and $(f, \varphi')$.

By virtue of (13), one obtains

$$(f, \varphi) = \sum_{k=1}^{t} \rho_k (\varphi, \lambda_k^2) \text{ and } (f, \varphi') = \sum_{k=1}^{t} \rho_k (\varphi', \lambda_k^2). \tag{7}$$

By virtue of (6), one obtains

$$(f, \varphi) = \sum_{h=1}^{\tau} \rho'_h (\varphi, {\lambda'_h}^2) \text{ and } (f, \varphi') = \sum_{h=1}^{\tau} \rho'_h (\varphi', {\lambda'_h}^2). \tag{8}$$

Let us notice that

$$(\varphi, \lambda_k^2) = M \text{ and } (\varphi', \lambda_k^2) \geq M; \;\; (k = 1, 2, \ldots, s) \tag{9}$$

$$(\varphi, {\lambda'_h}^2) \geq M \text{ and } (\varphi', {\lambda'_h}^2) = M. \;\; (h = 1, 2, \ldots, \sigma) \tag{10}$$

From equalities (7), one derives

$$(f, \varphi') - (f, \varphi) = \sum_{k=1}^{t} \rho_k \left[ (\varphi', \lambda_k^2) - (\varphi, \lambda_k^2) \right], \tag{11}$$

and by virtue of (3) and (9) there comes

$$(f, \varphi') - (f, \varphi) \geq 0.$$

From equalities (8), one derives

$$(f, \varphi') - (f, \varphi) = \sum_{h=1}^{\tau} \rho'_h \left[ (\varphi', {\lambda'_h}^2) - (\varphi, {\lambda'_h}^2) \right], \tag{12}$$

and by virtue of (6) and (10), one will have

$$(f, \varphi') - (f, \varphi) \leq 0.$$

It follows that

$$(f, \varphi') = (f, \varphi),$$

and the equalities (11) and (12) give

$$(\varphi', \lambda_k^2) = (\varphi, \lambda_k^2), \quad (k = 1, 2, \ldots, t)$$

$$(\varphi', {\lambda'_h}^2) = (\varphi, {\lambda'_h}^2). \quad (h = 1, 2, \ldots, \tau)$$

By virtue of (9) and (10), there arrive

$$(\varphi, {\lambda'_h}^2) = M, \quad (h = 1, 2, \ldots, \tau) \tag{13}$$

$$(\varphi', \lambda_k^2) = M. \quad (k = 1, 2, \ldots, t) \tag{14}$$

By virtue of equalities (13), the quadratic forms (5) are found among those of the series (1). By virtue of (14), the quadratic forms (2) are found among those of the series (4).

I argue that in this case the series (2) and (5) contain the same forms.

To demonstrate this, let us suppose that all the forms belonging to the face $P(\mu)$ verify the equations

$$\Psi_1(f) = 0,\ \Psi_2(f) = 0, \ldots,\ \Psi_r(f) = 0$$

and that any form belonging to the domain $R$ verifies the inequalities

$$\Psi_1(f) \geq 0,\ \Psi_2(f) \geq 0, \ldots,\ \Psi_r(f) \geq 0. \tag{15}$$

By virtue of (6), one will have

$$\rho'_1 \Psi_i({\lambda'_1}^2) + \rho'_2 \Psi_i({\lambda'_2}^2) + \ldots + \rho'_\tau({\lambda'_\tau}^2) = 0, \quad (i = 1, 2, \ldots, r)$$

and because of (15), one finds

$$\Psi_i({\lambda'_h}^2) = 0; \quad (i == 1, 2, \ldots, r; h = 1, 2, \ldots, \tau)$$

therefore all the forms of the series (5) belong to the series (2).

In the same way, one will demonstrate that all the forms of the series (2) belong to the series (5).

One concludes that the faces $P(\mu)$ and $P'(\nu)$ coincide, therefore the domains $R$ and $R'$ are contiguous through the face $P(\mu)$.

*Corollary. A quadratic form which is interior to a domain of the set* $(R)$ *can not belong to any other domain of that set.*

21

*Theorem. Let us suppose that to a face* $P(\mu)$ *of the domain* $R$ *belong positive quadratic forms. In this case, the number of domains of the set* $(R)$ *contiguous through the face* $P(\mu)$ *is finits.*

Let us indicate by

$$R, R_1, R_2, \ldots$$

the domains of the set $(R)$ contiguous through the face $P(\mu)$. Let

$$\varphi, \varphi_1, \varphi_2, \ldots$$

be the corresponding perfect forms having the minimum $M$.

On the ground of the supposition made, one positive quadratic form $f$ will be interior to the face $P(\mu)$.

We have demonstrated in the previous number that

$$(f,\varphi) = (f,\varphi_1) = (f,\varphi_2) = \dots \tag{16}$$

It is easy to demonstrate that the number of perfect forms having the minimum $M$ and verifying the equalities (16) is finite.

### *Algorithm for the search for the domain of the set $(R)$ contiguous to another domain by a face in $\frac{n(n+1)}{2} - 1$ dimensions*

22

Let

$$\varphi(x_1, x_2, \dots, x_n) = \sum a_{ij} x_i x_j$$

be a perfect form having the minimum $M$ the various representation of which make up a series

$$(l_{11}, l_{21}, \dots, l_{n1}), (l_{12}, l_{22}, \dots, l_{n2}), \dots, (l_{1s}, l_{2s}, \dots, l_{ns}). \tag{1}$$

Let us suppose that a face $P$ of the domain $R$ corresponding to the perfect form $\varphi$ be determined by the equation

$$\Psi(f) = \sum p_{ij} a_{ij} = 0$$

and by the condition

$$\Psi(f) \geq 0$$

which is verified by any quadratic form belonging to the domain $R$.

Let us suppose that the face $P$ be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \dots, \lambda_t^2 \tag{2}$$

where

$$\lambda_k = l_{1k} x_1 + l_{2k} x_2 + \dots + l_{nk} x_n. \quad (k = 1, 2, \dots, s)$$

On the ground of the supposition made, one will have the equalities

$$\sum p_{ij} l_{ik} l_{jk} = 0 \ \ (k = 1, 2, \dots, t) \tag{3}$$

which define the coefficients $P_{ij}$ $(i = 1, 2, \dots, n; j = 1, 2, \dots, n)$ to an immediate common factor.

Let us suppose that the face $P$ could belong to the other domains of the set $(R)$. Let us indicate by $R'$ a similar domain. Let $\varphi'$ be the perfect form corresponding to the domain $R'$.

By virtue of the supposition made, the quadratic form (2) belong to the domains $R$ and $R'$. It results in that the systems

$$(l_{11}, l_{21}, \dots, l_{n1}), (l_{12}, l_{22}, [\dots,] l_{n2}), \dots, (l_{1t}, l_{2t}, \dots, l_{nt}) \qquad (4)$$

corresponding to the forms (2) represent the minimum of forms $\varphi$ and $\varphi'$.

Let us suppose, for more simplicity, that the forms $\varphi$ and $\varphi'$ had the minimum $M$. One will have the equalities

$$\sum a_{ij} l_{ik} l_{jk} = M \text{ and } \sum a'_{ij} l_{ik} l_{jk} = M, \quad (k = 1, 2, \dots, t) \qquad (5)$$

by putting

$$\varphi'(x_1, x_2, \dots, x_n) = \sum a'_{ij} x_i x_j.$$

From equation (5), one gets

$$\sum (a'_{ij} - a_{ij}) l_{ik} l_{jk} = 0, \quad (k = 1, 2, \dots, t)$$

and by virtue of (3), it becomes

$$a'_{ij} = a_{ij} + \rho p_{ij}. \ (i = 1, 2, \dots, n; j = 1, 2, \dots, n) \qquad (6)$$

Let us indicate

$$\Psi(x_1, x_2, \dots, x_n) = \sum p_{ij} x_i x_j.$$

By virtue of (16), one obtains

$$\varphi'(x_1, x_2, \dots, x_n) = \varphi(x_1, x_2, \dots, x_n) + \rho \Psi(x_1, x_2, \dots, x_n). \qquad (7)$$

This stated, let us choose in the series (1) a system $(l_{1h}, l_{2h}, \dots, l_{nh})$ which does not belong to the series (4). As

$$\varphi(l_{1h}, l_{2h}, \dots, l_{nh}) = M \quad \varphi'(l_{1h}, l_{2h}, \dots, l_{nh}) \geq M$$

and

$$\Psi(l_{1h}, l_{2h}, \dots, l_{nh}) > 0,$$

one deduces from the equality

$$\varphi'(l_{1h}, l_{2h}, \dots, l_{nh}) = \varphi(l_{1h}, l_{2h}, \dots, l_{nh}) + \rho \Psi(l_{1h}, l_{2h}, \dots, l_{nh})$$

the inequality

$$\rho \geq 0.$$

The supposition $\rho = 0$ being obviously impossible, one obtains

$$\rho > 0,$$

and it follows that

$$\varphi'(l_{1h}, l_{2h}, \dots, l_{nh}) > M.$$

Let us indicate by

$$(l'_1, l'_2, \dots, l'_n), (l''_1, l''_2, \dots, l''_n), \dots, (l_1^{(r)}, l_2^{(r)}, \dots, l_n^{(r)}) \tag{8}$$

all the representations of the minimum of the perfect form $\varphi'$ which are not found in the series (4). By virtue of (7), one will have

$$\varphi'(l_1^{(k)}, l_2^{(k)}, \dots, l_n^{(k)}) = \varphi(l_1^{(k)}, l_2^{(k)}, \dots, l_n^{(k)}) + \rho\Psi(l_1^{(k)}, l_2^{(k)}, \dots, l_n^{(k)}) \qquad (k = 1, 2, \dots, r)$$

which results in

$$\varphi(l_1^{(k)}, l_2^{(k)}, \dots, l_n^{(k)}) > M \text{ and } \Psi(l_1^{(k)}, l_2^{(k)}, \dots, l_n^{(k)}) < 0. \qquad (k = 1, 2, \dots, r) \tag{9}$$

The value of the parameter $\rho$ will have for expression

$$\rho = \frac{\varphi(l_1^{(k)}, l_2^{(k)}, \dots, l_n^{(k)}) - 1}{-\Psi(l_1^{(k)}, l_2^{(k)}, \dots, l_n^{(k)})}. \quad (k = 1, 2, \dots, r)$$

Let us examine any one value of the function

$$\frac{\varphi(x_1, x_2, \dots, x_n) - M}{\Psi(x_1, x_2, \dots, x_n)} \tag{10}$$

determined with the condition

$$\Psi(x_1, x_2, \dots, x_n) < 0. \tag{11}$$

I argue that one will have the inequality

$$\frac{\varphi(x_1, x_2, \dots, x_n) - M}{\Psi(x_1, x_2, \dots, x_n)} \geq \rho.$$

Let us suppose the contrary. By supposing that

$$\frac{\varphi(x_1, x_2, \dots, x_n) - M}{\Psi(x_1, x_2, \dots, x_n)} < \rho,$$

one will find, because of (11),

$$\varphi(x_1, x_2, \dots, x_n) + \rho\Psi(x_1, x_2, \dots, x_n) < M$$

or, that which comes to the same thing because of (7),

$$\varphi'(x_1, x_2, \dots, x_n) < M,$$

which is contrary to the hypothesis.

We have arrived at the following important result:

*There exists only a single domain $R'$ contiguous to the domain $R$ through the face $P$. The corresponding perfect form $\varphi'$ will be determined by the equality (7) provided that the parameter $\rho$ presents the smallest positive value of the function (10).*

Let us notice that by virtue of (3) and (9), all the quadratic forms belonging to the domain $R'$ verify the inequality

$$\Psi(f) \leq 0.$$

One concludes that the domains $R$ and $R'$ are found from two opposite sides of the plane in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation

$$\Psi(f) = 0.$$

23

The smallest positive value of the function (10) can be obtained with the help of operations the number of which is finite.

The whole problem is reduced to the preliminary study of a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequality

$$\Psi(l_1, l_2, \ldots, l_n) < 0$$

and satisfying the condition that the quadratic form

$$\varphi_0(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_0 \Psi(x_1, x_2, \ldots, x_n),$$

where one has admitted

$$\rho_0 \frac{\varphi(l_1, l_2, \ldots, l_n) - M}{-\Psi(l_1, l_2, \ldots, l_n)}$$

be positive.

One will determine in this case all the systems $(x_1, x_2, \ldots, x_n)$ of integers verifying the inequality

$$\varphi(x_1, x_2, \ldots, x_n) \geq M \tag{12}$$

the number of which is finite, and one will find among these systems all those which define the smallest value of the function (10).

Let us indicate by $R$, as we have done in Number 2, the upper limit of values of the parameter $\rho$.

The problem is reduced to the study of a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequality

$$\varphi(l_1, l_2, \ldots, l_n) + R\Psi(l_1, l_2, \ldots, l_n) < M. \tag{13}$$

It can turn out that the equation

$$\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n) = 0$$

will be verified by integers, one will determine them with the help of equations

$$\frac{\partial\varphi}{\partial x_i} + R\frac{\partial\Psi}{\partial x_i} = 0. \quad (i = 1, 2, \ldots, n)$$

In the case where these equations can not be verified by any one system of integers, one will study the values of linear forms

$$\frac{\partial\varphi}{\partial x_i} + R\frac{\partial\Psi}{\partial x_i} \quad (i = 1, 2, \ldots, n)$$

and one will determine as many as one wish of the systems of integers verifying the inequality (13).

By supposing that a system of integers $(l_1, l_2, \ldots, l_n)$ verifying the inequality (13) were determined, one can look for the smallest positive value $\rho$ of the function (10) with the help of the following procedure.

The inequality (12) can be put under the form

$$\varphi(x_1, x_2, \ldots, x_n)\left(1 - \frac{\rho_0}{R}\right) + \frac{\rho_0}{R}\left[\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n)\right] \leq M,$$

and as

$$\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n) \geq 0,$$

it becomes

$$\varphi(x_1, x_2, \ldots, x_n)\left(1 - \frac{\rho_0}{R}\right) \leq M,$$

or differently

$$\varphi(x_1, x_2, \ldots, x_n) \leq M\frac{R}{R - \rho_0}.$$

Among the systems of integers verifying this inequality one will find all the systems (8) searched for.

*Algorithm for the search for the domain of the set $(R)$ to which belongs an arbitrary positive quadratic form.*

24

*Theorem. Any positive quadratic form belongs to at least one domain of the set $(R)$.*

Let

$$f(x_1, x_2, \ldots, x_n) = \sum a_{ij}x_ix_j$$

be any one positive quadratic form.

Let us choose any one domain $R$ from the set $(R)$.

Let us suppose that the form $f$, did not belong to the domain $(R)$.

In that case all the linear inequalities which defined the domain $R$ will not be verified. Let us suppose that one had the inequality

$$\Psi(f) = \sum p_{ij}a_{ij} < 0. \tag{1}$$

Let us indicate by $R_1$ the domain contiguous to the domain $R$ through the face in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation

$$\Psi(f) = 0.$$

By indicating with $\varphi$ and $\varphi_1$ the contiguous perfect forms corresponding to the domains $R$ and $R_1$, one will have, as we have seen in Number 22,

$$\varphi_1(x_1, x_2, \dots, x_n) = \varphi(x_1, x_2, \dots, x_n) + \rho\Psi(x_1, x_2, \dots, x_n) \tag{2}$$

where $\rho > 0$ and $\Psi(x_1, x_2, \dots, x_n) = \sum p_{ij}x_ix_j$.

Let us examine two results $(f, \varphi)$ and $(f, \varphi_1)$. By virtue of (2), one will have

$$(f, \varphi_1) = (f, \varphi) + \rho(f, \Psi),$$

and as, because of (1),

$$(f, \Psi) = \sum p_{ij}a_{ij} < 0,$$

it becomes

$$(f, \varphi) > (f, \varphi_1).$$

Let us suppose that by proceeding in this manner one obtains a series of domains

$$R, R_1, R_2, \dots. \tag{3}$$

By indicating with

$$\varphi, \varphi_1, \varphi_2, \dots \tag{4}$$

the series of corresponding perfect forms, one will have the inequalities

$$(f, \varphi) > (f, \varphi_1) > (f, \varphi_2) > \cdots$$

so long as the form $f$ did not belong to the domains (3).

By noticing that all the perfect forms (4) possess the same minimum $M$, one will easily demonstrate that the number of perfect forms (4) verifying the integrality

$$(f, \varphi) < P$$

is bounded, whatever may be the value of the positive parameter $P$.

One concludes that the series of domains (3) will necessarily be terminated by a domain $R_m$ to which belonged the form $f$ considered.

*Study of a complete system of domains representing the various classes of the set* ($R$).

25

Let $R$ be any one domain of the set $(R)$. Let us suppose that one had determined all the domains

$$R, R_1, R_2, \dots, R_\sigma \tag{1}$$

contiguous to the domain $R$ through the various faces in $\frac{n(n+1)}{2} - 1$ dimensions, then let us suppose that one had determined all the domains contiguous to the domains (1) and so on.

I say that by proceeding in this manner one will come across any domain of the set $(R)$ arbitrarily chosen.

For example, if one wish to arrive at a domain $R^{(0)}$, one will choose a positive quadratic form $f$ which is interior to the domain $R^{(0)}$ and one will proceed as we have done in Number 24. One will determine this way a series of domains

$$R, R', R'', \dots, R^{(\mu)}, R^{(0)}$$

which are successively contiguous through faces in $\frac{n(n+1)}{2} - 1$ dimensions.

We have seen in Number 19 that the set $(R)$ can be divided into classes of equivalent domains the number of which is finite.

Let us find a system of domains representing the various classes of the set $(R)$.

By starting from the domain $R$, we have determined all the domains

$$R_1, R_2, \dots, R_\sigma$$

contiguous to the domain $R$. By not considering equivalent domains as being different, let us choose among the domain (1) those which are not one to one equivalent and are not equivalent to the domain $R$. Let us suppose that one had obtained the series

$$R, R_1, R_2, \dots, R_{\mu-1} \tag{2}$$

of domains which are not one to one equivalent.

One will study in the same way the domains contiguous to the domains $R, R_1, R_2, \dots, R_{\mu-1}$ and one will extend the series (2) by adding to it new domains

$$R_\mu, R_{\mu+1}, \dots, R_{\nu-1}$$

which are not one to one equivalent and are not equivalent to the domains (2).

By proceeding in this way, one will always obtain a series

$$R, R_1, R_2, \dots, R_{\tau-1} \tag{3}$$

which enjoys the following property: the domain belonging to the series (3) are not one to one equivalent and all the domains contiguous to the domains (3) are equivalent to them.

The series (3) obtained presents a complete system of representations of various classes of the set $(R)$.

26

The study of the series (3) can be facilitated particularly by the help of substitutions which transform into itself the domains of the set $(R)$.

Let us suppose that the domain $R$ corresponding to a perfect form $\varphi$ be determined by the inequalities

$$\sum p_{ij}^{(k)} a_{ij} \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

By declaring

$$\Psi_k(x_1, x_2, \dots, x_n) = \sum p_{ij}^{(k)} x_i x_j, \quad (k = 1, 2, \dots, \sigma)$$

one will determine, as we have seen in Number 22, by the equalities

$$\varphi_k = \varphi + \rho_k \Psi_k \quad (k = 1, 2, \dots, \sigma) \tag{4}$$

$\sigma$ perfect forms $\varphi_1, \varphi_2, \dots, \varphi_\sigma$. One will call them contiguous to the perfect form $\varphi$.

Let us indicate by $g$ the group of substitutions which do not change the perfect form $\varphi$.

The perfect forms $\varphi_1, \varphi_2, \dots, \varphi_\sigma$ being well determined by the perfect form $\varphi$, one concludes that all the substitutions of the group $g$ will only permute the forms $\varphi_1, \varphi_2, \dots, \varphi_\sigma$.

By not considering as different the forms in proportional coefficients, one can say, by virtue of (4), that the group $g$ will only permute the quadratic forms

$$\Psi_1, \Psi_2, \dots, \Psi_\sigma. \tag{5}$$

Let us suppose that one had chosen in this series the forms

$$\Psi_1, \Psi_2, \dots, \Psi_{\mu-1} \tag{6}$$

which enjoyed the following properties: each form of the series (5) will be transformed into a form of the series (6) with the help of a substitution belonging to the group $g$, the forms (6) can not be transformed one to one with the help of substitutions of the group $g$.

The perfect forms

$$\varphi_k = \varphi + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \mu - 1)$$

can replace the perfect forms (4), therefore one will determine only the values of parameters $\rho_1, \rho_2, \ldots, \rho_{\mu-1}$.

The corresponding domains

$$R_1, R_2, \ldots, R_{\mu-1}$$

can replace the domains (1).

It can come to pass that among the domains $R, R_1, R_2, \ldots, R_{\mu-1}$ are found equivalent domains, one will recognise this with the help of particular methods.

*On a reduction method of positive quadratic forms.*

27

*Definition. One will call reduced any positive quadratic form belonging to any one domain*

$$R, R_1, R_2, \ldots, R_{\tau-1} \tag{1}$$

*of a complete system of representations of various classes of the set* $(R)$.

Let us suppose that one had determined all the substitutions

$$S_1, S_2, \ldots, S_m \tag{2}$$

which transform the domains contiguous with the domains (1) through the faces in $\frac{n(n+1)}{2} - 1$ dimensions into these domains here.

Let $f$ be any one positive quadratic form which is not reduced. One will determine with the help of the algorithm shown in Number 24 a series of domains

$$R, R', R'', \ldots, R^{(h)}$$

successively contiguous. Let us suppose that the domain $R^{(h)}$ be the first one which does not belong to the series (1).

With the help of a substitution $S'$ which is found among those of the series (2), one will transform the domain $R^{(h)}$ into a domain $R_k$ belonging to the series (1).

By transforming the form $f$ with the help of the substitution $S'$ into an equivalent form $f'$, one will determine with the help of the form $f'$ a new series of domains

$$R_k, R'_k, \ldots, R_k^{(t)}$$

and so on.

One will determine in this way a series of substitutions

$$S', S'', \ldots, S^{(\lambda)}$$

which, all, are found among those of the series (2) and the product

$$S = S'S'' \cdots S^{(\lambda)}$$

of which presents a substitution $S$ with the help of which the form $f$ will be transformed into a reduced form.

28

Let us suppose now that two reduced forms $f$ and $f'$ be equivalent.

If one of these forms, for example $f$, is interior to the domain $R_k$, the form $f'$ will also be interior to it. One concludes that the form $f$ can be transformed into a form $f'$ only with the help of a substitution which transforms the domain $R_k$ into itself.

Let us suppose that the reduced equivalent forms $f$ and $f'$ be interior to the faces in $\mu$ dimensions of domains (1).

In this case one will declare supplementary conditions for the reduced forms $f$ and $f'$. After having determined all the faces in $\mu$ dimensions of domains (1), one will choose a complete system of representatives of these various classes. Let us suppose that this system be formed by the faces in $\mu$ dimensions

$$P_1(\mu), P_2(\mu), \ldots, P_\nu(\mu). \tag{3}$$

Any positive quadratic form which is interior to a face in $\mu$ dimensions of a domain of the set $(R)$ will be equivalent to a form which is interior to the faces (3), one will call it reduced.

Two reduced positive quadratic forms which are interior to the faces (3) will be equivalent only provided that they be interior to the same face and that the substitution which transforms one of them into another one also transforms this face into itself.

We have arrived at the following result:

*A reduced quadratic form can be transformed into another reduced form or into itself only with the help of a substitution which transforms into itself a domain or a face of domains belonging to the series (1).*

## Second Part

### Some applications of the general theory to the study of perfect quadratic forms.

*On the principal perfect form*

29

We will not consider as different the quadratic forms of proportional coefficients, therefore one can arbitrarily choose the minimum value of a positive quadratic form.

In that which follows, one will study only the perfect quadratic forms whose minimum is 1. One will indicate by $D$ the determinant of these forms.

Among the various perfect forms, one form

$$\varphi = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n$$ [‡]

where

$$a_{ii} = 1, \ (i = 1, 2, \ldots, n), \ a_{ij} = \frac{1}{2}, \ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

and $D = \frac{n+1}{2^n}$

is distinguished by its simplicity.

One will call principal the perfect form $\varphi$.

The perfect form $\varphi$ possesses $\frac{n(n+1)}{2}$ representations of the minimum 1, which define $\frac{n(n+1)}{2}$ linear forms

$$\lambda_1 = x_1, \ \lambda_2 = x_2, \ldots, \ \lambda_n = x_n, \ \lambda_{n+1} = x_1 - x_2, \ \lambda_{n+2} = x_1 - x_3, \ldots, \ \lambda_{\frac{n(n+1)}{2}} = x_{n-1} - x_n.$$

The domain $R$ corresponding to the perfect form $\varphi$ is made up of all the quadratic forms determined by the equality

$$\sum a_{ij}x_ix_j = \sum_{k=1}^{\frac{n(n+1)}{2}} \rho_k \lambda_k^2 \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, \frac{n(n+1)}{2})$$

From this equality one obtains

$$\rho_k = a_{1k} + a_{2k} + \ldots + a_{nk} \text{ so long as } k = 1, 2, \ldots, n,$$

$$\rho_k = -a_{ij} \text{ so long as } k > n; \ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

---

The form $\varphi$ has been given for the first time by Zolotareff in a Mémoire titled: On an indeterminate equation of the third degree (in Russian) therefore the domain $R$ will be determined by the following inequalities:

$$\begin{cases} a_{1k}+a_{2k}+\ldots+a_{nk}\geq 0, \quad (k=1,2,\ldots,n) \\ -a_{ij}\geq 0. \quad (i=1,2,\ldots,n; j=1,2,\ldots,n; i\neq j) \end{cases} \tag{1}$$

By virtue of (1), the perfect form $\varphi$ possesses $\frac{n(n+1)}{2}$ contiguous perfect forms which are determined by the equalities

$$\begin{cases} \varphi_k=\varphi+\rho_k x_k(x_1,x_2,\ldots,x_n), \quad (k=1,2,\ldots,n) \\ \varphi_k=\varphi-\rho_k x_i x_j, \quad \big(k=n+1,n+2,\ldots,\frac{n(n+1)}{2}, i=1,2,\ldots,n; \\ \qquad\qquad j=1,2,\ldots,n; i\neq j) \end{cases} \tag{2}$$

30 Let us find equivalent forms among the perfect forms contiguous to the perfect form $\varphi$.

To this effect, let us determine the group $g$ of substitutions which do not change the form $\varphi$.

Let us examine, in the first place, the form adjointed to the form $\varphi$.

One will easily demonstrate that the coefficients of the form adjointed to $\varphi$ are proportional to those of the form

$$\omega=\lambda_1^2+\lambda_2^2+\ldots+\lambda_{\frac{n(n+1)}{2}}^2. \tag{3}$$

One concludes, by virtue of the theorem of Number 17, that the principal perfect form $\varphi$ is extreme.

The quadratic form $\omega$ will have for expression

$$\omega=nx_1^2+nx_2^2+\ldots+nx_n^2-2x_1x_2-2x_1x_3-\ldots-2x_{n-1}x_n$$

where

$$a_{ii}=n, \ (i=1,2,\ldots,n) \quad a_{ij}=-1. \ (i=1,2,\ldots,n; j=1,2,\ldots,n; i\neq j)$$

Let us find all the representations of the minimum of the form $\omega$.

The linear forms

$$x_1,x_2,\ldots,x_n,x_1+x_2+\ldots+x_n \tag{4}$$

characterise $n+1$ representations of the value $n$ of the form $\omega$.

I say that the form $\omega$ has the minimum $n$ and all the representations of the minimum of the form $\omega$ are characterised by the linear form (4).

To demonstrate this, let us examine any one value $\omega(x_1, x_2, \ldots, x_n)$ of the form $\omega$. By supposing that none of the numbers $x_1, x_2, \ldots, x_n$ becomes zero, one will have by virtue of (13)

$$\omega(x_1, x_2, \ldots, x_n) > n,$$

the system $x_1 = 1,\ x_2 = 1, \ldots, x_n = 1$ being excluded.

Let us suppose that any one of the numbers $x_1, x_2, \ldots, x_n$ does not cancel out and that

$$x_{k+1} = 0,\ \ x_{k+2} = 0, \ldots,\ \ x_n = 0;$$

one obtains, by virtue of (3),

$$\omega(x_1, x_2, \ldots, x_k, 0, \ldots, 0) = (n-k+1)(x_1^2+x_2^2+\ldots+x_k^2)+\sum(x_k-x_h)^2,$$

and it follows that

$$\omega(x_1, x_2, \ldots, x_n) \geq k(n-k+1),$$

therefore

$$\omega(x_1, x_2, \ldots, x_n) > n \quad \text{so long as} \quad k \geq 2.$$

This stated, let us indicate by $G$ the group of substitutions which transform into itself the domain $R$. By virtue of (3), any substitution of the group $G$ does not change the form $\omega$.

The group $g$ being adjointed to the group $G$, one concludes that each substitution of the group $g$ will only permute te linear forms (4) by changing the sign of a few among them.

By noticing that

$$x_1^2 + x_2^2 + \ldots + x_n^2 + (x_1 + x_2 + \ldots + x_n)^2 = 2\varphi,$$

one concludes that the group $g$ is composed of all the substitutions which permute the forms

$$x_1^2 + x_2^2 + \ldots + x_n^2 + (x_1 + x_2 + \ldots + x_n)^2.$$

Let us indicate

$$x_0 = -x_1 - x_2 - \ldots - x_n \ \text{ and } \ x'_0 = -x'_1 - x'_2 - \ldots - x'_n, \qquad (5)$$

and let $k_0, k_1, \ldots, k_n$ be any one permutation of numbers $0, 1, 2, \ldots, n$.

By posing

$$x_i = e_i x'_{k_i} \quad \text{where} \quad e_i = \pm 1,\ \ (i = 0, 1, 2, \ldots, n) \qquad (6)$$

one will have

$$x_0 + x_1 + \ldots + x_n = e_0 x'_{k_0} + e_1 x'_{k_1} + \ldots + e_n x'_{k_n},$$

and as, because of (5),

$$x_0 = x_1 + \ldots + x_n = 0 \quad \text{and} \quad x'_0, x'_1, \ldots, x'_n = 0,$$

it is necessary that

$$e_0 = e_1 = \ldots = e_n;$$

therefore the equalities (6) reduce to the one here:

$$x_i = ex'_{k_i}. \quad (i = 0, 1, 2, \ldots, n; e = \pm 1) \tag{7}$$

The number of substitutions defined by the formulae obtained is equal to $2 \cdot 1 \cdot 2 \cdots (n+1)$. By not considering as different two substitutions of opposite coefficients, one will say that the group $g$ is composed of $(n_1)!$ different substitutions. §

With the help of substitutions (7), one can transform any perfect form (2) contiguous to the principal form $\varphi$ into another form contiguous to the form $\varphi$, arbitrarily chosen.

We have arrived at the following important result.

*All the perfect forms contiguous to the principal perfect form are equivalent.*

31

Let us choose one form among those of the series (2). Let us declare

$$\varphi_1 = \varphi - \rho x_1 x_2.$$

All the perfect form contiguous to the form $\varphi$ are equivalent to the form $\varphi_1$.

Let us find the corresponding value of the parameter $\rho$.

As we have seen in Number 22, the value searched for of $\rho$ presents the smallest value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - 1}{x_1 x_2} \tag{8}$$

determined with condition

$$x_1 x_2 > 0.$$

One will distinguish in the subsequent studies two cases:

$$1). \ n = 2 \quad \text{and} \quad 2). \ n \geq 3.$$

*First case:*

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§ See: *Minkowski*, Zur Theorie der positiven quadratischen Formen [On the theory of the positive quadratic forms], (This Journal, V. 101, p. 200)

By comparing two $n = 2$ binary forms

$$\varphi = x_1^2 + x_2^2 + x_1x_2 \quad \text{and} \quad \varphi_1 = x_1^2 + x_2^2 + x_1x_2 - \rho x_1x_2,$$

one will notice that by making $\rho = 2$ one obtains the form

$$\varphi_1 = x_1^2 + x_2^2 - x_1x_2$$

which is evidently equivalent to the perfect form $\varphi$, therefore the perfect form $\varphi_1$ is that which one has searched for.

*Second case:*

By making

$$x_1 = 1,\ x_2 = 1,\ x_3 = -1,\ x_4 = 0, \ldots,\ x_n = 0,$$

one obtains a value of the function (8) which is equal to 1.

By making $\rho = 1$, one will present the form $\varphi_1$ under the following form:

$$\varphi_1 = \frac{1}{2}\left[(x_1, x_2, \ldots, x_n)^2 + (x_1 - x_2)^2 + x_3^2 + \ldots + x_n^2\right]. \qquad (9)$$

It results in that the form $\varphi_1$ is positive. On the ground of that which has been said in Number 23, one will find now all the systems of integers veifying the inequality

$$\varphi_1(x_1, x_2, \ldots, x_n) \leq 1.$$

By noticing that the inequality

$$\varphi_1(x_1, x_2, \ldots, x_n) < 1$$

is impossible, because the positive form $\varphi_1$ has integral values which corresponds to the integral values of variables, one concludes that the form $\varphi_1$ is perfect.

With the help of the equality (9), one will easily determine all the presentations of the minimum of the perfect form $\varphi_1$.

*On the binary and ternary perfect forms and on the domains which correspond to them.*

32

The binary principal perfect form

$$\varphi = x_2 + xy + y_2, \quad D = \frac{3}{4}$$

possesses, as we have seen in Number 29, three contiguous perfect forms which are equivalent to the principal form.

One concludes that all the perfect binary forms constitute only a single class of forms equivalent to the principal form.

The domain $\mathcal{R}$ corresponding to the principal form is made up of binary forms $(a, b, c)$ which are determined by the equality

$$ax^2 + 2bxy + cy^2 = \rho x^2 + \rho' y^2 + \rho''(x - y)^2$$

where

$$\rho \geq 0, \;\; \rho' \geq 0, \;\; \rho'' \geq 0$$

It follows that the domain $R$ is determined by the inequalities

$$\rho = a + b \geq 0, \;\; \rho' = -b \geq 0, \;\; \rho'' = c + b \geq 0.$$

By calling reduced the positive binary forms verifying these inequalities, as we have done in Number 27, one will establish a well known method of reduction, due to Mr. Selling. ¶

It results in that the domain $R^0$ is determined by the inequalities

$$\rho = c - a \geq 0, \;\; \rho' = a + 2b \geq 0, \;\; \rho'' = -b \geq 0.$$

The inequalities obtained only differ from famous conditions of reduction of positive binary quadratic forms due to Lagrange by the choice of the sign of the coefficient $b$, that which one can arbitrarily make in the method of Lagrange. ‡

33

Let us examine now the ternary perfect perfect forms.

The principal perfect form

$$\varphi = x_2 + y^2 + z^2 + yz + zx + xy, \quad D = \frac{1}{2}$$

possesses six contiguous perfect forms which, all, are equivalent to the perfect form

$$\varphi_1 = x_2 + y^2 + z^2 + yz + zx$$

which we have found in Number 31.

The substitution

$$x = -x', \; y = y', \; z = -y' - z'$$

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¶ *Selling.* Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p. 143)

‡ See: *Lagrange.* Recherches d'Arithmétique. [Studies in arithmetic] (Oeuvres de Lagrange published by Serret, V. III, p. 698)

*Gauss.* Disquisitiones arithmeticae, art. 171. (Werke, V. I.)

*Lejeune Dirichlet.* Vorlesungen über Zahlentheorie [Letcures on number theory], published by Dedekind, (Braunschweig 1894, §64, p. 155) transforms the form $\varphi_1$ into principal form.

One concludes that all the ternary perfect forms form only a single class.

The domain $R$ corresponding to the principal form is made up of all the ternary quadratic forms $\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix}$ which are determined by the equality

$$\begin{aligned} ax^2{+}a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = \\ \rho_1' + \rho_2 y^2 + \rho_3 z^2 + \rho_4(y-z)^2 + \rho_5(z-x)^2 + \rho_6(x-y)^2. \end{aligned}$$

The domain $R$ is determined by the inequalities

$$\begin{array}{lll} \rho_1 = a + b' + b'' \geq 0, & \rho_2 = a' + b'' + b \geq 0, & \rho_3 = a'' + b + b' \geq 0, \\ \rho_4 = -b \geq 0, & \rho_5 = -b' \geq 0, & \rho_6 = -b'' \geq 0. \end{array}$$

By calling reduced the positive ternary quadratic forms belonging to the domain $R$, one will establish a method of reduction due to Mr. Selling.

The domain $R$ can be partitioned into 24 equivalent parts which can be transformed one into another with the help of 24 substitution adjoined to those which do not change the principal form.

One of these parts, the domain $\mathcal{R}$, will be composed of all the ternary quadratic forms determined by the equality

$$\begin{aligned} ax^2{+}a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = \\ \rho_1 x^2 + \rho_2 y^2 + \rho_3 z^2 + \rho_4(y-z)^2 + \rho_5 \Psi + \rho_{[6]}\omega \end{aligned}$$

where

$$\Psi = x^2{+}y^2{+}z^2(y{-}z)^2{+}(z{-}x)^2,\ \omega = x^2{+}y^2{+}z^2(y{-}z)^2{+}(z{-}x)^2{+}(x{-}y)^2.$$

One will determine the domain $\mathcal{R}$ with the help of inequalities

$$\begin{array}{lll} \rho_1 = a + 2b' + b'' \geq 0, & \rho_2 = a' + b + b' + b'' \geq 0, & \rho_3 = a'' + b + b' + b'' \geq 0, \\ \rho_4 = -b + b' \geq 0, & \rho_5 = -b' + b'' \geq 0, & \rho_6 = -b'' \geq 0. \end{array}$$

The domain $\mathcal{R}$ enjoys the following properties:

1. Any positive ternary quadratic form is equivalent to at least one form belonging to the domain $\mathcal{R}$.

2. Two ternary quadratic forms which are interior to the domain $\mathcal{R}$ can not be equivalent.

By effecting the transformation of the domain $\mathcal{R}$ with the help of all the substitutions of integral coefficients and of determinant ±1, one will make up the set $(\mathcal{R})$ of domains.

Each domain $\mathcal{R}$ belonging to the set $(\mathcal{R})$ possesses six domain contiguous by faces in 5 dimensions.

The domain $\mathcal{R}$ will be transformed into contiguous domain with the help of the following substitutions:

$$S_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix},$$
$$S_4 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Each substitution of this series transforms into itself a corresponding face in 5 dimensions of the domain $\mathcal{R}$ and permutes two domains of the set $(\mathcal{R})$ which are contiguous through this face.

This results in a method for the search for the substitution which transforms a given form into a form belonging to the domain $\mathcal{R}$. This method is analogous to that which has been shown in Number 27.

By calling reduced any positive ternary quadratic form belonging to the domain $\mathcal{R}$, one will establish a new method of reduction of positive ternary quadratic forms which can be considered as a generalisation of the method of reduction of Lagrange.

34 *On the perfect form* $x_1^2+x_2^2+\ldots+x_n^2+x_1x_3+x_1x_4+\ldots+x_{n-1}x_n$.

Let us examine the perfect form

$$\varphi_1 = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_3 + x_1x_4 + \ldots + x_{n-1}x_n$$

obtained in Number 31. One has admitted

$$a_{ii} = 1, \ (i = 1, 2, \ldots, n), \ \ a_{12} = 0, \ \ a_{ij} = \frac{1}{2}.$$
$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

It results in that

$$D = \frac{1}{2^{n-2}}.$$

By supposing that $n \geq 4$, one will have $n^2 - n$ representations of the minimum of the form $\varphi_1$ the number of which is greater than $\frac{n(n+1)}{2}$.

These representations of the minimum of the form $\varphi_1$ will be characterised by the linear forms

$$\begin{cases} \lambda_1 = x_1, \lambda_2 = x_2, \ldots, \lambda_n = x_n, \lambda_{n+1} = x_1 - x_3, \ldots, \\ \lambda_{\frac{n(n+1)}{2}-1} = x_{n-1} - x_n,\ \lambda_{\frac{n(n+1)}{2}} = x_1 + x_2 - x_3, \ldots, \\ \lambda_{\frac{n(n+1)}{2}+n-3} = x_1 + x_2 - x_n, \lambda_{\frac{n(n+1)}{2}+n-2} = x_1 + x_2 - x_3 - x_4, \\ \quad \ldots, \lambda_{n^2-n} = x_1 + x_2 - x_{n-1} - x_n. \end{cases} \tag{1}$$

The domain $R_1$ corresponding to the perfect form $\varphi_1$ is made up of forms deterined by the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n^2-n} \rho_k \lambda_k^2 \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, [n^2 - n])$$

Let us find the linear inequalities which define the domain $R_1$.

The number of these inequalities is so large in deed for $n = 4$.

One will overcome the difficulties which result by the help of a particular method.

35

Let us find the group $g_1$ of substitutions which do not change the form $\varphi_1$.

To this effect, let us introduce in our studies a quadratic form $\omega$ determined by the equality

$$\omega = \frac{2}{n-1}(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{n^2-n}^2).$$

After the reductions, one obtains

$$\begin{aligned} \omega(x_1, x_2, \ldots, x_n) =& nx_1^2 + nx_2^2 + 4x_3^2 + \ldots + 4x_n^2 + 2(n-2)x_1x_2 - \\ & 4x_1x_3 - \ldots - 4x_1x_n - 4x_2x_3 - \ldots - 4x_2x_n. \end{aligned}$$

One can give in the form $\omega_2$ the following expression:

$$\omega(x_1, x_2, \ldots, x_n) = (x_1 - x_2)^2 + (x_1 + x_2)^2 + (x_1 + x_2 - 2x_3)^2 + \ldots + (x_1 + x_2 - 2x_n)^2.$$

It is easy to demonstrate that the form added to the perfect form $\varphi_1$ has coefficients which are proportional to those of the form $\omega$.

It follows that the perfect form $\varphi_1$ is extreme.

Let us observe that the linear form

$$x_1 + x_2 + \ldots + x_n,\ x_1 - x_2,\ x_3,\ x_4, \ldots,\ x_n$$

characterise $n$ minimum 4 representations of the form $\omega$. In the case $n \geq 5$, other representations of the minimum of the form $\omega$ do not exist;

in the case $n = 4$, one obtains 12 representations of the minimum of the form $\omega$.

By noticing that

$$\varphi_1 = \frac{1}{2}\left[(x_1 + x_2 + \ldots + x_n)^2 + (x_1 - x_2)^2 + x_3^2 + \ldots + x_n^2\right],$$

one can say that the group $g_1$, in the case $n \geq 5$, is composed of all the permutations of the forms

$$(x_1 + x_2 + \ldots + x_n)^2,\ (x_1 - x_2)^2,\ x_3^2, \ldots,\ x_n^2.$$

In the case $n = 4$, one will determine by this method only divisor of the group $g_1$.

By indicating

$$\begin{array}{llll} u_1 = x_1 + x^2 + \ldots + x_n, & u_2 = x_1 - x_2, & u_3 = x_3, & \ldots, \quad u_n = x_n, \\ u_1' = x_1' + x_2' + \ldots + x_n', & u_2' = x_1' - x_2', & u_3' = x_3', & \ldots, \quad u_n' = x_n' \end{array}$$

let us declare

$$u_i = e_i u'_{k_i}, \quad (i = 1, 2, \ldots, n) \tag{2}$$

where $e_i = \pm 1$ $(i = 1, 2, \ldots, n)$ and the indices $k_1, k_2, \ldots, k_n$ present any one permutation of numbers $1, 2, \ldots, n$.

Each system of equalities (2) defines a substitution of the group $g_1$.

One concludes that the group $g_1$ is composed of $2^{n-1}n!$ different substitutions, in the case $n \geq 5$.

36

Let us suppose that the domain $R_1$ be determined by the inequalities

$$\sum p_{ij}^{(k)} a_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

By indicating

$$\Psi_k(x_1, x_2, \ldots, x_n) = \sum p_{ij}^{(k)} x_i x_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine, as we have seen in Number 22, $\sigma$ perfect forms

$$\varphi_1^{(k)} = \varphi_1 + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \sigma) \tag{3}$$

contiguous to the perfect form $\varphi_1$.

All the substitutions of the group $g_1$ will make only one permutation of forms

$$\Psi_1, \Psi_2, \ldots, \Psi_\sigma. \tag{4}$$

Let us effect the transformation of forms (3) and (4) with the help of the sustitution

$$x_1, x_2, \ldots, x_n = x'_1,\ x_1 - x_2 = x'_2,\ x_3 = x'_3, \ldots,\ x_n = x'_n. \qquad (5)$$

The series (4) will be transformed into a series

$$\Psi'_1, \Psi'_2, \ldots, \Psi'_\sigma.$$

Let us indicate by **g** a group of substitutions

$$x'_i = e_i x''_{k_i}, \quad (i = 1, 2, \ldots, n)$$

where $e = \pm 1$ $(i = 1, 2, \ldots, n)$ and $k_1, k_2, \ldots, k_n$ present a permutation of numbers $1, 2, \ldots, n$. Each substitution of the group **g** makes only one permutation of forms (6), and to a similar substitution corresponds a substitution of the group $g_1$.

By indicating

$$\Psi'_k(x'_1, x'_2, \ldots, x'_n) = \sum P^{(k)}_{ij} x'_i x'_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine with the help of inequalities

$$\sum P^{(k)}_{ij} a_i a_j \geq 0, \quad (k = 1, 2, \ldots, \sigma) \qquad (7)$$

a domain $\mathcal{R}$.

The form $\varphi_1$ will be transformed into a form

$$\frac{1}{2}({x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2),$$

with the help of the substitution (5), and any system $(x_1, x_2, \ldots, x_n)$ of integers $x_1, x_2, \ldots, x_n$ will be replaced by a system $(x'_1, x'_2, \ldots, x'_n)$ of number, also integral, $x'_1, x'_2, \ldots, x'_n$ satisfying the condition

$$x'_1 + x'_2 + \ldots + x'_n \equiv 0 (\mathrm{mod}\ 2). \qquad (8)$$

It results in that the linear forms (1) which correspond to the various representations of the minimum of the form $\varphi_1$ will be replaced by the forms

$$x'_i + x'_j \ \text{ and } \ x'_i - x'_j \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

which characterise the various representations of the minimum 2 of the quadratic form ${x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2$, in the set $(X')$ of all the systems $(x'_1, x'_2, \ldots, x'_n)$ of integers $x'_1, x'_2, \ldots, x'_n$ satisfying the condition (8).

One concludes that the edges of the domain $\mathcal{R}$ will be characterised by the quadratic form

$$(x'_i + x'_j)^2 \quad \text{and} \quad (x'_i - x'_j)^2. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

By virtue of (7), one obtains the inequalities

$$P_{ii}^{(k)} + 2P_{ij}^{(k)} + P_{jj}^{(k)} \geq 0 \text{ and } P_{ii}^{(k)} - 2P_{ij}^{(k)} + P_{jj}^{(k)} \geq 0. \qquad (9)$$
$$(k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

Let us examine any one form

$$\Psi'(x'_1, x'_2, \ldots, x'_n) = \sum P_{ij} x'_i x'_j \qquad (10)$$

belonging to the series (6). By virtue of (9), one will have

$$P_{ii} + 2P_{ij} + P_{jj} \geq 0 \text{ and } P_{ii} - 2P_{ij} + P_{jj} \geq 0. \qquad (11)$$
$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

Among these conditions one will find $t$ quantities which define the coefficients of the form (10) to an immediate common factor. All these equalities will be of the form

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1. \qquad (12)$$

Let us suppose that there exists a combination of values of $k$ and $h$ satisfying the conditions

$$P_{kk} + 2P_{kh} + P_{hh} > 0 \text{ and } P_{kk} - 2P_{kh} + P_{hh} > 0. \qquad (13)$$

By noticing that the coefficient $P_{kh}$ does not enter the other inequalities (11), one concludes that the coefficient $P_{kh}$ remains undetermined.

For all the coefficients of the form (10) to be determined by the conditions (12) to an immediate common factor, it is necessary, the coefficient $P_{kh}$ being independent of other coefficients, that all the coefficients which remain cancel out.

By virtue of inequalities (13), this supposition is impossible, therefore the inequality (12) has to hold for all the values of indices $k$ and $h$.

One obtains $\frac{n(n-1)}{2}$ conditions

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1. \qquad (14)$$
$$(k = 1, 2, \ldots, n; h = 1, 2, \ldots, n; k \neq h)$$

which serve to determine the coefficients $P_{kh}$ in functions of coefficients

$$P_{11}, P_{22}, \ldots, P_{nn}. \qquad (15)$$

The coefficients $P_{11}, P_{22}, \ldots, P_{nn}$ can not be independent, and will be connected by at least $n-1$ equations of the form (12). Therefore, in at least $n-1$ case, one will have the equations of the form

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0. \qquad (16)$$

To make short we will call these equations double.

This stated, let us suppose, in the first place, that there exists at least one coefficient among those of the series (15) which does not enter in the double equations (16). One can suppose, to fix the ideas, that $P_{11}$ be such a coefficient.

The coefficient $P_{11}$ being independent, all the coefficients $P_{22}, \ldots, P_{nn}$ will cancel each other and, by virtue of (14), the coefficients

$$P_{23}, P_{24}, \ldots, P_{n-1,n}$$

will also cancel one another out.

The coefficient $P_{11}$ is used for determining the coefficients $P_{12}$, $P_{13}, \ldots, P_{1n}$ with the help of equations (14) which take the form

$$P_{11} - 2e_{1k}P_{1k} = 0; \quad (k = 2, 3, \ldots, n)$$

it follows that

$$2P_{1k} = e_{1k}P_{11}. \quad (k = 2, 3, \ldots, n)$$

As, on the ground of the supposition made,

$$P_{11} + 2e_{1k}P_{1k} > 0, \quad (k = 2, 3, \ldots, n)$$

it is necessary that

$$P_{11} > 0,$$

and one can declare

$$P_{11} = 1.$$

The form (10) is determined by the equalities obtained, and one will have

$$\Psi'(x'_1, x'_2, \ldots, x'_n) = {x'_1}^2 + e_{12}x'_1x'_2 + \ldots + e_{1n}x'_1x'_n. \qquad (17)$$

By replacing the variables

$$e_{12}x'_2, \ e_{13}x'_3, \ldots, \ e_{1n}x'_n$$

by the variables $x'_2, \ldots, x'_n$, one will replace the form (17) by the form

$$\Psi'(x'_1, x'_2, \ldots, x'_n) = x'_1(x'_1, x'_2, \ldots, x'_n).$$

Let us suppose, in the second place, that all the coefficients (15) enter in the double equations (16).

At least one of the coefficients (15) is not zero. Let us suppose that $P_{kk} \neq 0$. Following the hypothesis, the coefficient $P_{kk}$ enters in at least one double equation

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0.$$

It follows that

$$P_{kh} = 0 \text{ and } P_{kk} + P_{hh} = 0,$$

therefore the coefficients $P_{kk}$ and $P_{hh}$ are of opposite signs. Let us suppose, to fix the ideas, that

$$P_{11} = -1. \tag{18}$$

By examining the inequalities

$$P_{11} \pm 2P_{1k} + Pkk \geq 0, \quad (k = 2, 3, \dots, n)$$

one deduces

$$Pkk > 0. \quad (k = 2, 3, \dots, n)$$

It results in that the double equation

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0$$

has to be impossible so long as $k \geq 2$ and $h \geq 2$, therefore all the double equations will be of the form

$$P_{11} \pm 2P_{1k} + P_{kk} = 0. \quad (k = 2, 3, \dots, n)$$

From these equations one gets, by virtue of (18),

$$P_{kk} = 1 \text{ and } P_{1k} = 0. \quad (k = 2, 3, \dots, n) \tag{19}$$

By substituting the values obtained of coefficients $P_{11}, P_{22}, \dots,$ $P_{nn}$ in the equations

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1, \quad {\scriptstyle (k = 2, 3, \dots, n;\, h = 2, 3, \dots, n;\, k \neq h)}$$

one obtains, because of (19),

$$P_{kh} = e_{kh} \quad \text{where} \quad e_{kh} = \pm 1. \; {\scriptstyle (k = 2, 3, \dots, n;\, h = 2, 3, \dots, n;\, k \neq h)}$$

The form (10) will have for expression

$$\begin{aligned} \Psi'(x_1', x_2', \dots, x_n') = &- x_1'^2 + x_2'^2 + x_3'^2 + \dots + x_n'^2 + 2e_{23}x_2'x_3' \\ &+ 2e_{24}x_2'x_4' + \dots + 2e_{n-1,n}x_{n-1}'x_n' \end{aligned} \tag{20}$$

where

$$e_{23} = \pm 1, \; e_{24} = \pm 1, \dots, \; e_{n-1,n} = \pm 1.$$

One obtains in this way $2^{\frac{(n-1)(n-2)}{2}}$ different forms. By permuting the variables and by changing their signs, one will particularly decrease the number of various forms determined by the formula (20).

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With the help of results obtained, one can easily recognise whether a given quadratic form $\sum a_{ij}x_ix_j$ belongs to the domain $\mathcal{R}$ or not.

One will examine, in the first place, the sums

$$e_{1k}a_{1k}+e_{2k}a_{2k}+\ldots+e_{nk}a_{nk} \quad \text{where } e_{1k}=\pm1, e_{2k}=\pm1, \ldots, e_{nk}=\pm1 \text{ and } e_{kk}=1. \quad (k=1,2,\ldots,n)$$

All these sums have to be positive or zero. The inequalities

$$a_{kk}-|a_{1k}|-\ldots-|a_{k-1,k}|-|a_{k+1,k}|-\ldots-|a_{nk}| \geq 0, \quad (k=1,2,\ldots,n) \tag{21}$$

present the conditions necessary and sufficient for the inequalities

$$e_{1k}a_{1k}+e_{2k}a_{2k}+\ldots+e_{nk}a_{nk} \geq 0 \quad (k=1,2,\ldots,n)$$

to hold.

Let us examine, in the second place, the inequalities

$$-a_{11}+a_{22}+a_{33}+\ldots+a_{nn}+2e_{23}a_{23}+2e_{24}a_{24}+\ldots+2e_{n-1,n}a_{n-1,n} \geq 0$$

where

$$e_{23}=\pm1, \quad e_{24}=\pm1, \ldots, \quad e_{n-1,n}=\pm1,$$

These inequalities can be replaced by a single one

$$-a_{11}+a_{22}+a_{33}+\ldots+a_{nn}-2|a_{23}|-2|a_{24}|-\ldots-2|a_{n-1,n}| \geq 0.$$

One will present this inequality under the form

$$a_{11}+a_{22}+\ldots+a_{nn}-2|a_{12}|-2|a_{13}|-\ldots -2|a_{n-1,n}| \geq 2\left(a_{11}-|a_{12}|-\ldots-|a_{1n}|\right).$$

By permuting the variables, one obtains $n$ inequalities

$$a_{11}+a_{22}+\ldots+a_{nn}-2|a_{12}|-2|a_{13}|-\ldots -2|a_{n-1,n}| \geq 2\left(a_{kk}-|a_{1k}|-\ldots-|a_{nk}|\right). \text{where } k=1,2,\ldots,n \tag{22}$$

We have arrived at the following result. One can easily recognise whether a given positive quadratic form $f$ belong to the domain $R_1$ or not. To this effect, one will transform the form $f$ by a form $f'$ with the help of the substitution adjointed to the substitution (5) and one will examine $2n$ inequalities (21) and (22). For the form $f$ to belong to the domain $R_1$, it is necessary and sufficient that the form $f'$ verifies $2n$ inequalities (21) and (22).

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Let us return now to the perfect forms (3) contiguous to the perfect form $\varphi_1$. We have seen that these forms will be transformed with the help of the substitution (5) into forms

$$\frac{1}{2}\left({x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2\right) + \rho_k \Psi'_k(x'_1, x'_2, \ldots, x'_n). \quad (k = 1, 2, \ldots, \sigma)$$

The forms $\Psi'_1, \Psi'_2, \ldots, \Psi'_\sigma$ can be transformed with the help of substitutions belonging to the group $\mathbf{g}$ into forms

$$\left\{\begin{aligned} &1).\ x'_3(-x'_1 - x'_2 + x'_3 + x'_4 + \ldots + x'_n), \\ &2).\ -{x'_2}^2 + {x'_1}^2 + {x'_3}^2 + \ldots + {x'_n}^2 - 2x'_1x'_3 - \ldots \\ &\qquad\qquad - 2x'_1x'_n + 2e_{34}x'_3x'_4 + \ldots + 2e_{n-1,n}x'_{n-1}x'_n, \end{aligned}\right. \tag{23}$$

where

$$e_{34} = \pm 1, \ldots,\ e_{n-1,n} = \pm 1.$$

The inverse substitution to the substitution (5):

$$x'_1 = x_1 + x_2 + \ldots + x_n,\ x'_2 = x_1 - x_2,\ x'_3 - x_3, \ldots,\ x'_n = x_n$$

will transform the forms (23) into forms

$1).\ -2x_1x_3$

$2).4(x_1x_2 - \delta_{34}x_3x_4 - \ldots - \delta_{n-1,n}x_{n-1}x_n)$, where; $\delta_{34} = 0$ or $1, \ldots,$ $\delta_{n-1,n} = 0$ or $1$.

One concludes that all the perfect forms contiguous to the form $\varphi_1$ are equivalent to the following perfect forms

$$\begin{aligned} &1).\varphi_1 - \rho x_1x_3, \\ &2).\varphi_1 + \rho(x_1x_2 - \delta_{34}x_3x_4 - \ldots - \delta_{n-1,n}x_{n-1}x_n), \end{aligned}$$

where

$$\delta_{34} = 0 \text{ or } 1, \ldots,\ \delta_{n-1,n} = 0 \text{ or } 1.$$

*Study of the perfect form* $\varphi_1 - \rho x_1x_3$.

39 The perfect form $\varphi_1$, possesses, as we have seen in Number 38, many contiguous perfect forms which are not equivalent.

One will determine in the following only a single perfect form

$$\varphi_2 = \varphi_1 - \rho x_1x_3$$

contiguous to the perfect form $\varphi_1$.

We have demonstrated in Number 22 that the parameter $\rho$ presents the smallest value of the function

$$\rho = \varphi_1 \frac{(x_1, x_2, \ldots, x_n) - 1}{x_1x_3} \tag{1}$$

determined on condition that

$$x_1x_3 > 0. \tag{2}$$

By declaring

$$x_1 = 1,\ x_2 = 0,\ x_3 = 1,\ x_4 = -1,\ x_5 = 0, \dots,\ x_n = 0,$$

one obtains the value of the function (1) which is equal to 1, therefore

$$0 < \rho \leq 1. \tag{3}$$

Let us effect the transformation of the function (1) with the help of the substitution

$$x_3 = x'_1,\ -x_1+x_2 = x'_2,\ x_1+x_2+\dots+x_n = -x'_3,\ x_4 = x'_4, \dots,\ x_n = x'_n \tag{4}$$

one will have

$$\rho = \frac{{x'_1}^2 + {x'_2}^2 + \dots + {x'_n}^2 - 2}{-x'_1(x'_1 + x'_2 + \dots + x'_n)} \tag{5}$$

where, because of (2),

$$x'_1(x'_1 + x'_2 + \dots + x'_n) < 0 \tag{6}$$

and, because of (4),

$$x'_1 + x'_2 + \dots + x'_n \equiv 0 (\mathrm{mod}\ 2),$$

the variables $x'_1 + x'_2 + \dots + x'_n$ being integers.

Let us indicate

$$f(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2 + \rho x_1(x_1 + x_2 + \dots + x_n).$$

By virtue of (5) and (6) the value looked for of $\rho$ is defined by the conditions that the inequality

$$f(x_1, x_2, \dots, x_n) < 2$$

is impossible, so long as the integers $x_1, x_2, \dots, x_n$ verify the congruence

$$x_1 + x_2 + \dots + x_n \equiv 0 (\mathrm{mod}\ 2), \tag{7}$$

and that there exists at least one system $(l_1, l_2, \dots, l_n)$ verifying the equation

$$f(x_1, x_2, \dots, x_n) = 2 \tag{8}$$

and the congruence (7).

The form $f$ can be determined by the equality

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = \left(x_2 + \rho\frac{x_1}{2}\right)^2 + \left(x_3 + \rho\frac{x_1}{2}\right)^2 + \ldots \\ + \left(x_n + \rho\frac{x_1}{2}\right)^2 + \left(1 + \rho - \frac{n-1}{4}\rho^2\right) x_1^2. \end{aligned} \tag{9}$$

It follows that the form $f$ will be positive, provided that

$$1 + \rho - \frac{n-1}{4}\rho^2 > 0,$$

and the upper limit $R$ of values of $\rho$ verifies the equation

$$1 + R - \frac{n-1}{4}R^2 = 1,$$

therefore

$$\frac{R = 2}{\sqrt{n} - 1.} \tag{10}$$

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This presented, let us examine a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the equation (8) and the congruence (7).

I say that there will be the inequalities

$$\left| l_i + \rho\frac{l_1}{2} \right| \leq 1. \quad (i = 2, \ldots, n) \tag{11}$$

In effect, if one suppose that

$$\left| l_k + \rho\frac{l_1}{2} \right| < 1,$$

one will determine $e_k = \pm 1$ such that the inequality

$$\left| l_k + 2e_k + \frac{l_1}{2} \right| < \left| l_k + \rho\frac{l_1}{2} \right|$$

holds, and one will present

$$l'_i = l_i \text{ and } l'_k = l_k = 2e_k. \quad (i = 1, 2, \ldots, n; i \neq k)$$

The condition (7) will be satisfied, and one will have, by virtue of (9), the inequality

$$f(l'_1, l'_2, \ldots, l'_n) < 2,$$

which is contrary to the hypothesis.

By examining the inequalities (11) and the form $f$ with the help of the formula (9), one will easily demonstrate that among the system of integers verifying the equation (8) with condition (7) is found at least one system $(l_1, l_2, \dots, l_n)$ satisfying the conditions

$$f(l_1, l_2, \dots, l_n) = 2 \tag{12}$$

and

$$l_2 = l_3 + \delta,\ l_3 = l_4 = \dots = l_n \ \text{ where } \ \delta = 0 \ \text{ or } \ \pm 1. \tag{13}$$

By virtue of (6), one will have the inequality

$$l_1 \left[ l_1 + \delta + (n-2) l_3 \right] < 0.$$

One can suppose that

$$l_1 < 0, \tag{14}$$

and it follows that

$$l_1 + \delta + (n-2) l_3 > 0,$$

therefore, because of (13) and (14), it is necessary that

$$l_3 > 0.$$

I say that $l_3 = 1$. To demonstrate this, let us effect the transformation of the positive quadratic form $f(x_1, x_2, \dots, x_n)$ with the help of the substitution

$$x_1 = -x,\ x_2 = y,\ x_3 = x_4 = \dots = x_n - z; \tag{15}$$

one will obtain a ternary positive form

$$F(x, y, z) = x^2 + y^2 + (n-2) z^2 - \rho x(-x + y + (n-2) z).$$

By virtue of the condition (7), the integers $x, y, z$ verify the congruence

$$x + y + (n-2) z \equiv 0 (\text{mod } 2). \tag{16}$$

By indicating

$$u = -l_1, \quad v = l_2, \quad w = l_3,$$

one will have, because of (12), (13) and (15),

$$F(u, v, w) = 2,$$

and the condition (16) will be fulfilled.

The inequality

$$f(x, y, z) < 2$$

is impossible so long as the integers $x, y, z$ verify the congruence (16).

Let us effect the transformation of the form $F(x,y,z)$ with the help of the substitution

$$x = x_{,} + y' + (n-2)z', \quad y = x' - y', \quad z = z'. \tag{17}$$

The set of systems $(x,y,z)$ of integers verifying the congruence (16) will be replaced by the set of systems $(x',y',z')$ of arbitrary integers.

Let us indicate by $F'(x',y',z')$ the transformed form. Let $D$ and $D'$ be the determinants of forms $F(x,y,z)$ and $F'(x',y',z')$. By virtue of (17), one will have

$$D' = 4D. \tag{18}$$

Let us notice that the number 2 presents the minimum of the form obtained $F'(x',y',z')$ determined in the set of all the systems $(x',y',z')$ of integers, the system $(0,0,0)$ being excluded.

On the ground of the known theorem § on the limit of the minimum of a ternary positive quadratic form, one will have the inequality

$$2 \leq \sqrt[3]{2D'}.$$

It follows that

$$D' \geq 4,$$

and because of (18), one obtains

$$D \geq 1. \tag{19}$$

This presented, let us observe that the form $F(x,y,z)$ has the following values:

$$F(u,v,w) = 2, \quad F(1,1,0) = 2, \quad F(1,-1,0) = 2 + 2\rho.$$

By transforming the form $F(x,y,z)$ with the help of the substitution

$$\begin{pmatrix} u, & 1, & 1 \\ v, & 1, & -1 \\ w, & 0, & 0 \end{pmatrix}, \tag{20}$$

one obtains a form

$$F_0(x',y',z') = ax'^2 + a'y'^2 + a''z'^2 + 2by'z' + 2b'z'x' + 2b''x'y',$$

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§ See: *Gauss.* Werke, V. II, p. 192, Göttingen 1863.

*Lejeune-Dirichlet.* Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen. (This Journal, V. 40, p. 209)

*Hermite.* Sur la théorie des formes quadratiques ternaires. [On the theory of ternary quadratic forms] (This Journal, V. 40, p. 173) where

$$a = 2,\ a' = 2,\ a'' = 2 + 2\rho \text{ and } b = \rho. \tag{21}$$

The product $a \cdot a' \cdot a''$ in any positive ternary quadratic form $\begin{pmatrix} a, & a', & a'' \\ b, & b' & b'' \end{pmatrix}$ is, as one knows, always greater than the determinant of the form, unless the coefficients $b, b', b''$ do not simultaneously cancel one another out.

By indicating with $D_0$ the determinant of the form $F_0(x', y', z')$, one will have, because of (21),

$$D_0 < 4(2 + 2\rho),$$

and as, by virtue of (20),

$$D_0 = 4w^2 D,$$

it becomes

$$w^2 D < 2 + 2\rho.$$

By virtue of (3) and (19), one obtains the inequality

$$w^2 < 4,$$

therefore

$$w = 1.$$

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By returning to the equalities (13), one obtains

$$l_1 = -u,\ \ l_2 = \delta \text{ and } l_3 = 1,\ \ l_4 = 1, \ldots,\ \ l_n = 1,$$

where

$$u > 0 \quad \text{and} \quad \delta = 0, 1, 2.$$

By substituting the values found of $l_1, l_2, \ldots, l_n$ in the function (5), one will have

$$\rho = \frac{u^2 + \delta^2 + n - 4}{u(-u + \delta + n - 2)} \tag{22}$$

It remains to determine the smallest value of this function providing that

$$u > 0,\ -u + \delta + n - 2 > 0,\ u \equiv n + \delta (\mathrm{mod}\ 2) \text{ and } \delta = 0, 1, 2. \tag{23}$$

Let us admit

$$u = \sqrt{n} - 1 + \alpha, \tag{24}$$

$\alpha$ being a real number.

The function (22) takes the form

$$\rho = \frac{2n + (2\alpha - 2)\sqrt{n} + \alpha^2 - 2\alpha + \delta^2 - 3}{\sqrt[n]{n} + (\alpha - 2)n + (\delta - 2\alpha)\sqrt{n} + 1 - \alpha^2 + \alpha\delta - \delta}.$$

The value searched for of $\rho$ has to verify the inequality

$$\rho < R,$$

therefore because of (10), one will have

$$\frac{2}{\sqrt{n} - 1} - \rho > 0. \tag{25}$$

After the reductions, one obtains

$$\frac{2}{\sqrt{n} - 1} - \rho = \frac{(1 - \delta^2 + 2\delta - \alpha^2)\sqrt{n} + \delta^2 - 2\delta - 1 - \alpha^2 - 2\alpha + 2\alpha\delta}{(\sqrt{n} - 1)\left[n\sqrt{n} + (\alpha - 2)n + (\delta - 2\alpha)\sqrt{n} + 1 - \alpha^2 + \alpha\delta - \delta\right]}$$

and, because of (25), it becomes

$$(1 - \delta^2 + 2\delta - \alpha^2)\sqrt{n} + \delta^2 - 2\delta - 1 - \alpha^2 - 2\alpha + 2\alpha\delta > 0.$$

By noticing that

$$\delta^2 - 2\delta - 1 - \alpha^2 - 2\alpha + 2\alpha\delta \leq 0 \quad \text{so long as} \quad \delta = 0, 1, 2,$$

one obtains the inequality

$$1 - \delta^2 + 2\delta - \alpha^2 > 0.$$

By making $\delta = 0$ and 2, one will have

$$\alpha^2 < 1 \ \text{ as long as } \ \delta = 0 \ \text{ and } \ 2 \tag{26}$$

By making $\delta = 1$, one will have

$$\alpha^2 < 2 \quad \text{so long as} \quad \delta = 1. \tag{27}$$

Let us indicate by $m$ a positive integer determined with the help of inequalities

$$\sqrt{n} - 1 \leq m < \sqrt{n}. \tag{28}$$

By declaring

$$n = m^2 + p, \tag{29}$$

one will have a positive integer $p$ verifying the inequalities

$$0 < p \leq 2m + 1. \tag{30}$$

*First case:* $p$ is an odd number.

By virtue of (23) and (29), one will have a congruence

$$u \equiv m^2 + p + \delta (\text{mod } 2),$$

$p$ being an odd number; one can declare

$$u = m^2 + \delta + 1 + 2t. \tag{31}$$

By declaring

$$\sqrt{n} = m + \xi,$$

one will have

$$0 < \xi \leq 1, \tag{32}$$

because of (28). By virtue of (24), one obtains the equality

$$u = m - 1 + \xi + \alpha,$$

and because of (31), it becomes

$$\xi + \alpha = m^2 - m + 2 + 2t + \delta. \tag{33}$$

By supposing that $\delta = 0$ or 2, one obtains

$$\xi + \alpha \equiv 0 (\text{mod } 2).$$

By virtue of (26) and (32), it is necessary that

$$\xi + \alpha = 0,$$

therefore

$$u = m - 1 \;\text{ so long as }\; \delta = 0 \;\text{ and }\; \delta = 2.$$

By supposing that $\delta = 1$, one obtains, because of (33),

$$\xi + \alpha \equiv 1 (\text{mod } 2).$$

By virtue of (27) and (32), the integer $\xi + \alpha$ can have only two values

$$\xi + \alpha = \pm 1,$$

and it results in that

$$u = m \quad \text{or} \quad m - 2 \quad \text{as long as} \quad \delta = 1.$$

One obtains four values of the function (22):

$$\rho_1 = \frac{(m-1)^2+n-4}{(m-1)(n-m-1)}, \qquad \rho_2 = \frac{m^2+n-3}{m(n-m-1)},$$
$$\rho_3 = \frac{(m-2)^2+n-3}{(m-2)(n-m+1)}, \quad \rho_4 = \frac{(m-2)^2+n}{(m-1)(n-m+1)}$$

among which is found the smallest value looked for of $\rho$.

By noticing that

$$\rho_1 - \rho_2 = \frac{p-3}{m(m-1)(n-m-1)},$$

$$\rho_4 - \rho_1 = \frac{2p+2}{(m-1)(n-m-1)(n-m+1)},$$

$$\rho_3 - \rho_4 = \frac{p+1}{(m-1)(m-2)(n-m+1)},$$

one obtains, because of (30),

$$\rho_1 < \rho_4 < \rho_3$$

and

$$\rho_2 \leq \rho_1 \quad \text{so long as} \quad p \geq 3,$$

$$\rho_1 < \rho_2 \quad \text{so long as} \quad p < 3.$$

There exists only a single odd value of $p$ verifying the inequalities $0 < p < 3$, therefore one will have the inequality

$$\rho_1 < \rho_2 \quad \text{so long as} \quad p = 1.$$

We have arrived at the following result.

*The smallest value of* $\rho$ *will have for expression*

$$\rho = \frac{m^2+n-3}{m(n-m-1),} \tag{34}$$

*provided that* $n = m^2+p$, *and the odd number* $p$ *verifies the inequalities*

$$3 \leq p \leq 2m+1.$$

*In the case* $n = m^2+1$, *the smallest value of* $\rho$ *will be*

$$\rho = \frac{(m-1)^2+n-4}{(m-1)(n-m-1)}.$$

*Second case:* $p$ is an even number.

One will have, because of (23), the inequality [sic]

$$u \equiv m^2 + \delta (\mathrm{mod}\ 2).$$

By presenting

$$u = m^2 + \delta + 2t,$$

one will have the equalities

$$u = m - 1 + \xi + \alpha \quad \text{and} \quad \xi + \alpha = m^2 - m + 2t + \delta + 1.$$

By supposing that $\delta = 0$ or 2, one obtains

$$\xi + \alpha = 1,$$

and it follows that

$$u = m \quad \text{so long as} \quad \delta = 0 \quad \text{and} \quad 2.$$

By supposing that $\delta = 1$, one obtains

$$\xi + \alpha = 0 \quad \text{or} \quad \xi + \alpha = 2,$$

therefore

$$u = m + 1 \quad \text{or} \quad u = m + 1 \quad \text{so long as} \quad \delta = 1.$$

The smallest value of $\rho$ is found among the following values of the function (22):

$$\rho_1 = \tfrac{m^2+n-4}{m(n-m-2)}, \quad \rho_2 = \tfrac{(m+1)^2+n-3}{(m+1)(n-m-2)},$$
$$\rho_3 = \tfrac{(m-1)^2+n-3}{(m-1)(n-m)}, \quad \rho_4 = \tfrac{m^2+n}{m(n-m)}.$$

By noticing that

$$\rho_2 - \rho_1 = \frac{2m + 4 - p}{m(m+1)(n-m-2)},$$
$$\rho_1 - \rho_4 = \frac{4m - 2p}{m(n-m)(n-m-2)},$$
$$\rho_4 - \rho_3 = \frac{2m - p}{m(m-1)(n-m)},$$

one obtains, because of (30),

$$\rho_3 \leq \rho_4 \leq \rho_1 < \rho_2.$$

We have arrived at the following result:

*The smallest value of* $\rho$ *is expressed by the equality*

$$\rho = \frac{(m-1)^2 + n - 3}{(m-1)(n-m)}$$

*provided that* $n = m^2 + p$, *and the even number* $p$ *verifies the inequalities*

$$0 < p < 2m + 1.$$

We have determined the value of the parameter $\rho$ which defines the perfect form $\varphi_1 + \rho x_1 x_3$. The determinant $D$ of this form, by virtue of (4) and (9), will have for expression

$$D = \frac{4 + 4\rho - (n-1)\rho^2}{2^n}. \tag{35}$$

The corresponding value of the function $\mathcal{M}(a_{ij})$ defined in Number 16 will be

$$\mathcal{M}(a_{ij}) = 2\sqrt[n]{\frac{1}{4 + 4\rho - (n-1)\rho^2}}.$$

By applying the formulae obtained to the case:

$$n = 4, 5, 6, 7, 8,$$

one obtains the same value of $\rho$

$$\rho = 1.$$

The corresponding perfect forms will be

$$x_1^2 + x_2^2 + \ldots, x_4^2 + x_1 x_4 + \ldots + x_3 x_4,\ D = \frac{5}{2^4},\ \mathcal{M}(a_{ij}) = 2\sqrt[4]{\frac{1}{5}};$$
$$x_1^2 + x_2^2 + \ldots, x_5^2 + x_1 x_4 + \ldots + x_4 x_5,\ D = \frac{4}{2^5},\ \mathcal{M}(a_{ij}) = 2\sqrt[5]{\frac{1}{4}};$$
$$x_1^2 + x_2^2 + \ldots, x_6^2 + x_1 x_4 + \ldots + x_5 x_6,\ D = \frac{3}{2^6},\ \mathcal{M}(a_{ij}) = 2\sqrt[6]{\frac{1}{3}};$$
$$x_1^2 + x_2^2 + \ldots, x_7^2 + x_1 x_4 + \ldots + x_6 x_7,\ D = \frac{2}{2^7},\ \mathcal{M}(a_{ij}) = 2\sqrt[7]{\frac{1}{2}};$$
$$x_1^2 + x_2^2 + \ldots, x_8^2 + x_1 x_4 + \ldots + x_7 x_8,\ D = \frac{1}{2^8},\ \mathcal{M}(a_{ij}) = 2.$$

One comes across all these perfect forms in the Mémoire of Mr.'s Korkine and Zolotareff: Sur les formes quadratiques. [On the quadratic forms] ‡

The formulae obtained give a mean for the study of various perfect forms which verify the inequality

$$\mathcal{M}(a_{ij}) > 2.$$

By making, for example, $n = 12$, one will have

$$m = 3 \quad \text{and} \quad p = 3.$$

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‡ Mathematische Annalen, V. VI, p. 367.

By virtue of (34), one obtains

$$\rho = \frac{3}{4},$$

therefore, because of (35),

$$D = \frac{13}{16} \cdot \frac{1}{2^{12}},$$

and it follows that

$$\mathcal{M}(a_{ij}) = 2\sqrt[12]{\frac{16}{13}} > 2.$$

All the extreme forms studied by Mr.'s Korkine and Zolotareff do not give a function $\mathcal{M}(a_{ij})$ of values which exceed 2.

*On the quadratic perfect forms and on the domains which correspond to them.*

43

We have seen in Number 29 that to the quaternary principal perfect form

$$\varphi = x_1^2 + x_2^2 + x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, \quad D = \frac{5}{2^4}$$

corresponds the domain $R$ made up of forms

$$\rho_1 x_1^2 + \rho_2 x_2^2 + \rho_3 x_3^2 + \rho_4 x_4^2 + \rho_5 (x_1 - x_2)^2 + \rho_{[6]} (x_1 - x_3)^2 + \rho_7 (x_1 - x_4)^2$$
$$+ \rho_8 (x_2 - x_3)^2 + \rho_9 (x_2 - x_4)^2 + \rho_{10} (x_3 - x_4)^2.$$

All the perfect forms contiguous to the principal form $\varphi$ are equivalent to the form

$$\varphi_1 = x_1^2 + x_2^2, x_3^2, x_4^2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, \quad D = \frac{1}{4}.$$

The corresponding domain $R_1$ is made up of forms

$$\rho_1 x_1^2 + \rho_2 x_2^2 + \rho_3 x_3^2 + \rho_4 x_4^2 + \rho_5 (x_1 - x_3)^2 + \rho_6 (x_1 - x_4)^2$$
$$+ \rho_7 (x_2 - x_3)^2 + \rho_8 (x_2 - x_4)^2 + \rho_9 (x_3 - x_4)^2 + \rho_{10} (x_1 + x_2 - x_3)^2$$
$$+ \rho_{11} (x_1 + x_2 - x_4)^2, \rho_{12} (x_1 + x_2 - x_3 - x_4)^2.$$

Let us examine the perfect forms contiguous to the perfect form $\varphi_1$.

We have demonstrated in Number 38 that all these forms are equivalent to the forms

1). $\varphi_1 - \rho x_1 x_3,$

2). $\varphi_1 + \rho(x_1 x_2 - \delta x_3 x_4),$ where $\delta = 0$ or $1$.

Let us examine three perfect forms

$$1).\ \varphi_1 + \rho x_1 x_2, \quad 2).\ \varphi_1 - \rho x_1 x_3, \quad 3).\ \varphi_1 + \rho(x_1 x_2 - x_3 x_4).$$

1). By making $\rho = 1$ in the form $\varphi_1 + \rho x_1 x_2$, one obtains the principal perfect form $\varphi$.

2). Let us notice that the form $\varphi_1 - \rho x_1 x_3$ is equivalent to the form $\varphi_1 + \rho x_1 x_2$.

In effect, the substitution

$$x_1 = -x_1',\ x_2 = x_3',\ x_3 = x_2',\ x_4 = x_1' + x_4'$$

does not change the form $\varphi_1$ and transforms the form $x_1 x_2$ into the form $-x_1' x_3'$.

3). By making $\rho = 1$ in the form $\varphi_1 + \rho(x_1 x_2 - x_3 x_4)$, one obtains the form

$$x_1^2 + x_2^2, x_3^2 + x_4^2 + x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4$$

which is evidently equivalent to the perfect form $\varphi_1$.

One concludes that all the perfect forms contiguous to the perfect form $\varphi_1$ are equivalent to the forms $\varphi$ and $\varphi_1$.

It follows that the set of all the quaternary perfect forms be divided into two classes represented by the perfect forms $\varphi$ and $\varphi_1$.

The set $(R)$ of domains corresponding to various quaternary perfect forms is made up of two classes, too, represented by the domains $R$ and $R_1$.

*On the perfect forms in five variables and on the domains which correspond to them.*

44 We have determined two perfect forms in five variables

$$\varphi = x_1^2 + x_2^2 + \ldots + x_5^2 + x_1 x_2 + x_1 x_3 + \ldots + x_4 x_5, \quad D = \frac{6}{2^5},$$

$$\varphi_1 = x_1^2 + x_2^2 + \ldots + x_5^2 + x_1 x_3 + x_1 x_4 + \ldots + x_4 x_5, \quad D = \frac{4}{2^5}.$$

The corresponding domains $R$ and $R_1$ will be composed of forms

$$R)\ \rho_1 x_1^2 + \rho_2 x_2^2 + \ldots + \rho_5 x_5^2 + \rho_6 (x_1 - x_2)^2 + \rho_7 (x_1 - x_3)^2 + \ldots + \rho_{15} (x_4 - x_5)^2,$$

$$R_1)\ \rho_1 x_1^2 + \rho_2 x_2^2 + \ldots + \rho_5 x_5^2 + \rho_6 (x_1 - x_3)^2 + \ldots + \rho_{20} (x_1 + x_2 - x_4 - x_5)^2.$$

Examine the perfect forms contiguous to the perfect face $\varphi_1$. We have demonstrated in Number 38 that all these forms are equivalent to the forms

$$\begin{aligned} &1).\quad \varphi_1 - \rho x_1 x_3, \\ &2).\quad \varphi_1 - \rho(x_1x_2 - \delta x_3x_4 - \delta' x_3x_5 - \delta'' x_4x_5). \end{aligned} \tag{1}$$

where

$$\delta = 0 \text{ or } 1, \quad \delta' = 0 \text{ or } 1, \quad \delta'' = 0 \text{ or } 1.$$

In the second case one obtains 8 perfect forms. By permuting the variables $x_3, x_4, x_5$ one will replace the forms (1) by 4 forms; thus all the perfect forms contiguous to the perfect form $\varphi_1$ are equivalent to the 5 following forms:

$$\begin{gathered} 1).\ \varphi_1 + \rho x_1x_2, \quad 2).\ \varphi_1 + \rho x_1x_3, \quad 3).\ \varphi_1 + \rho(x_1x_2 - x_4x_5), \\ 4).\ \varphi_1 + \rho(x_1x_2 - x_3x_5 - x_4x_5), \quad 5).\ \varphi_1 + \rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5). \end{gathered}$$

1). By making $\rho = 1$ in the perfect form $\varphi_1 + \rho x_1x_2$, one obtains the perfect form $\varphi$.

2). We have seen in Number 42 that the perfect form $\varphi_1 - \rho x_1x_3$ is determined by the value $\rho = 1$ of the parameter $\rho$ in the case $n = 5$. One obtains the form

$$\varphi_1' = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_1x_4 + x_1x_5 + \ldots + x_4x_5 \tag{2}$$

which will be transformed with the help of the substitution

$$x_1 = -x_2^2,\ x_2 = x_1' - x_2',\ x_3 = x_3',\ x_4 = x_2 + x_4',\ x_5 = x_2' + x_5'$$

into a perfect form $\varphi_1$.

3). In the form $\varphi_1 + \rho(x_1x_2 - x_4x_5)$, one will put $\rho = 0$ and one will obtain the form

$$x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_3x_5$$

which is evidently equivalent to the form $\varphi_1$.

4). In the form $\varphi_1 + \rho(x_1x_2 - x_3x_5 - x_4x_5))$, one will put $\rho = 1$ and one will obtain the form

$$x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_2x_5$$

which is evident to the perfect form (2)

5). It remains only to determine the perfect form:

$$\varphi_1 + \rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5). \tag{3}$$

By effecting the transformation with the help of the substitution

$$-x_1+x_2=x'_1,\ \ x_1+x_2+x_3+x_4+x_5=x'_2,\ \ x_3=x'_3,\ \ x_4=x'_4,\ \ x_5=x'_5 \tag{4}$$

of the form

$$2\varphi_1+2\rho(x_1x_2-x_3x_4-x_3x_5-x_4x_5),$$

one obtains the form

$$\begin{aligned}{x'_1}^2+{x'_2}^2+{x'_3}^2+{x'_4}^2+{x'_5}^2+\frac{\rho}{2}\Big[&-{x'_1}^2+{x'_2}^2+{x'_3}^2+{x'_4}^2+{x'_5}^2 \\ &-2x'_2x'_3-2x'_2x'_4-2x'_2x'_5-2x'_3x'_4-2x'_3x'_5-2x'_4x'_5\Big].\end{aligned} \tag{5}$$

By virtue of (4) the integral variables $x'_1, x'_2, x'_3, x'_4, x'_5$ verify the congruence

$$x'_1+x'_2+x'_3+x'_4+x'_5\equiv 0(\mathrm{mod}2). \tag{6}$$

By applying to the form (5) the method unveiled in Number 2, one will determine the value of the upper limit $R>0$ of value $\frac{\rho}{2}$ with the help of equations

$$\xi_1-R\xi_1=0,\ \xi_2+R(\xi_2-\xi_3-\xi_4-\xi_5)=0,\ \xi_3+R(-\xi_2+\xi_3-\xi_4-\xi_5)=0,$$
$$\xi_4+R(-\xi_2-\xi_3+\xi_4-\xi_5)=0,\ \xi_5+R(-\xi_2-\xi_3-\xi_4+\xi_5)=0.$$

It results in that

$$\xi_2=\xi_3=\xi_4=\xi_5,$$

and one obtains the equations

$$\xi_1(1-R)=0 \quad \text{and} \quad \xi_2(1-2R)=0,$$

thus

$$R=\frac{1}{2}.$$

By declaring

$$x'_1=0,\ x'_2=1,\ x'_3=1,\ x'_4=1, x'_5=1, \tag{7}$$

one will satisfy the condition (6) and one will have the value $4-4\rho$ of the form (5).

By making

$$4-4\rho=2,$$

one obtains

$$\rho=\frac{1}{2}.$$

It follows that the positive quadratic form

$$x_1'^2 + x_2'^2 + \ldots + x_5'^2 + \frac{1}{4}\left[-x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 + x_5'^2 - 2x_2'x_3' - \ldots - 2x_4'x_5'\right] \tag{8}$$

will have a value 2 corresponding to the system (7).

By virtue of that which has been discussed in Number 23, the smallest value of the form (8) will correspond to a system $(l_1, l_2, \ldots, l_5)$ verifying the inequality

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 \leq 2 \cdot \frac{R}{R - \frac{1}{4}} \quad \text{where} \quad R = \frac{1}{2}.$$

One obtains the inequality

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 \leq 4.$$

It is easy to demonstrate that the system (7) is the only one verifying this inequality on condition (6), the systems which verify the inequality

$$-x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 + x_5'^2 - 2x_2'x_3' - 2x_2'x_4' - 2x_2'x_5' - 2x_3'x_4' - 2x_3'x_5' - 2x_4'x_5' \geq 0$$

being excluded. By making $\rho = \frac{1}{2}$ in the form (3), one obtains the perfect form

$$\varphi_2 = x_1' + x_2' + \ldots + x_5' + \frac{1}{2}x_1x_2 + x_1x_3 + \ldots + x_2x_5 + \frac{1}{2}x_3x_4 + \frac{1}{2}x_3x_5 + \frac{1}{2}x_4x_5,$$

$$D = \left(\tfrac{3}{2}\right)^4 \tfrac{1}{2^5}.$$

The corresponding domain $R^2$ is composed of forms

$$\begin{gathered}\rho_1 x_1^2 + \rho_2 x_2^2 + \ldots + \rho_5 x_5^2 + \rho_6(x_1 - x_3)^2 + \ldots + \rho_{11}(x_2 - x_5)^2 \\ +\rho_{12}(x_1 + x_2 - x_3 - x_4)^2 + \rho_{13}(x_1 + x_2 - x_3 - x_5)^2 \\ +\rho_{14}(x_1 + x_2 - x_4 - x_5)^2 + \rho_{15}(-x_1 - x_2 + x_3 + x_4 + x_5)^2.\end{gathered}$$

The number of parameter $\rho_1, \rho_2, \ldots, \rho_{15}$ being equal to the number of dimensions of the domain $R_2$, one will determine without trouble 15 inequalities which define the domain $R_2$.

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We have demonstrated that all the perfect forms contiguous to the perfect form $\varphi_1$ are equivalent to the perfect forms $\varphi, \varphi_1$ and $\varphi_2$.

Choose the perfect forms contiguous to the perfect form $\varphi_2$.

To this effect let us notice, in the first place, that the perfect form $\varphi_1$ is contiguous to the perfect form $\varphi_2$, then observe that all the perfect forms contiguous to the form $\varphi_2$ are equivalent.

To demonstrate this, examine all the faces in 14 dimensions of the domain $R_2$.

The domain $R_2$ is characterised by 15 quadratic forms

$$\left\{\begin{array}{l} x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_1 - x_5)^2, (x_2 - x_3)^2, \\ (x_2 - x_4^2), (x_2 - x_5)^2, (x_1 + x_2 - x_3 - x_4)^2, (x_1 + x_2 - x_3 - x_5)^2, \\ (x_1 + x_2 - x_4 - x_5)^2, (-x_1 - x_2 + x_3 + x_4 + x_5)^2. \end{array}\right. \tag{9}$$

Each face in 14 domains of the domain $R_2$ possesses 14 of these, and the form which remains can be called form opposite to the face.

One concludes that each face is well determined by the opposite face.

For the perfect forms contiguous to the perfect form $\varphi_2$ to be equivalent, it is necessary and sufficient that all the faces of the domain $R_2$ could be transformed one to one with the help of substitutions which do not change the domain $R_2$.

It would be easy to write all these substitutions, but one will proceed in another way, more speedy.

Let us observe that the face $P$ belonging to the domain $R_1$ and $R_2$ is characterised by all the forms (9), the form $(-x_1-x_2+x_3+x_4+x_5)^2$ being excluded.

With the aid of substitution associated with the substitution (4), one will replace the forms (9) by the forms:

$$\left\{\begin{array}{l} (x'_1 \pm x'_2)^2, (x'_1 \pm x'_3)^2, (x'_1 \pm x'_4)^2, (x'_1 \pm x'_5)^2, \\ (x'_2 + x'_3)^2, (x'_2 + x'_4)^2, (x'_2 + x'_5)^2, (x'_3 + x'_4)^2, (x'_3 + x_{[}5]')^2, \\ (x'_4 + x'_5)^2, (x'_2 + x'_3 + x'_4 + x'_5)^2. \end{array}\right. \tag{10}$$

By changing the sign of $x'_1$ and by permuting the variables $x'_2$, $x'_3$, $x'_4$, $x'_5$, one will transform into itself the forms (10), and the form $(x'_2 + x'_3 + x'_4 + x'_5)^2$ will not change.

To each similar substitution corresponds a substitution which transforms into itself the domain $R_2$ and the face $P$ of the domain $R_2$, and does not change the form $(-x_1 - x_2 + x_3 + x_4 + x_5)^2$.

By changing the sign of $x'_1$ and by permuting $x'_2, x'_3, x'_4, x'_5$, one will transform the form $(x'_1 + x'_2)^2$ into forms

$$(x'_1 \pm x'_2)^2, (x'_1 \pm x'_3)^2, (x'_1 \pm x'_4)^2, (x'_1 \pm x'_5)^2$$

and one will transform the form $(x'_2 + x'_3)^2$ into forms

$$(x'_2 + x'_3)^2, (x'_2 + x'_4)^2, (x'_2 + x'_5)^2, (x'_3 + x'_4)^2, (x'_3 + x'_{[5]})^2, (x'_4 + x'_5)^2.$$

Thus only the forms

$$(x'_1 + x'_2)^2,\ (x'_2 + x'_3)^2,\ (x'_2 + x'_3 + x'_4 + x'_5)^2 \tag{11}$$

remain to examine.

By returning to the forms (9), one obtains the forms corresponding to the forms (11).

$$x_2^2, x_3^2, (-x_1 - x_2 + x_3 + x_4 + x_5)^2. \tag{12}$$

It is demonstrated that all the forms (9) can be transformed into forms (12) with the help of substitutions which do not change the domain $R_2$.

With the help of substitutions

$$x_1 = x_2' - x_5',\ x_2 = x_3',\ x_3 = x_2',\ x_4 = x_1' + x_2',\ x_4 = x_1' + x_2' - x_4' - x_5',$$
$$x_5 = -x_1' + x_3'$$

and

$$x_1 = x_1',\ x_2 = -x_1' - x_2' + x_3' + x_4' + x_5',\ x_3 = x_3',\ x_4 = x_4',\ x_5 = x_5',$$

one will transform the domain $R_2$ into itself, and the form $x_2^2$ will be transformed into forms $x_3^2$ and $(-x_1' - x_2' + x_3' + x_4' + x_5')^2$.

We have demonstrated that all the forms of the domain $R_2$ are equivalent. It results in, from that we have seen, that all the perfect forms contiguous to the perfect form $\varphi_2$ are equivalent to the perfect form $\varphi_1$.

One concludes that all the perfect forms in five variables constitute three different classes represented by the perfect forms $\varphi, \varphi_1$ and $\varphi_2$.

The set of domains $(R)$ can be divided into three classes also, represented by the domains $R, R_1$, and $R_2$.

End of the first Mémoire.

New applications of continuous parameters to the theory of quadratic forms

# Second Memoir

## Research on the primitive parallelohedron

by Mr. Georges Voronoï in Warsaw

[*Journal für die reine und angewandte Mathematik*, V. 134, 1908]

[translated by K N Tiyapan]

The well known method of reduction for the binary, ternary and quaternary positive quadratic forms † rests upon a property of the positive quadratic form, to know:

*Every positive quadratic form* $\sum_{i=1}^{n}\sum_{i=1}^{n}a_{ij}x_ix_j$ *has n variables in the set E composing all of the systems* $(x_1, x_2, \ldots, x_n)$ *of integers of the variables* $x_1$, $x_2$, …, $x_n$ *n consecutive minima*

$$M_1 \leq M_2 \leq \cdots \leq M_n$$

*determined at condition which the determinant* $\omega$ *of a system*

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1n}, l_{2n}, \ldots, l_{nn}) \qquad (1)$$

*which represent these minima in the set E does not vanish.*

In all the cases where one has

$$\omega = \pm 1$$

one can transform the quadratic form $\sum\sum a_{ij}x_ix_j$ into an equivalent form by using a substitution

$$x_i = \sum_{k=1}^{n} l_{ik}x'_k \qquad (i = 1, 2, \ldots, n)$$

In the transformed form $\sum\sum a'_{ij}x_ix_j$, one will have

$$a'_{kk} = M_k \qquad (k = 1, 2, \ldots, n)$$

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† *Lagrange,* Recherches d'Arithmétique [Studies in arithmetic] (*Oeuvres,* V. III, p. 695)

*Gauß,* Disquisitiones arithméticae (*Oeuvres,* V. I, art. 171, p. 146)

*Lejeune-Dirichlet,* Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen [On the reduction of the positive quadratic forms with three indeterminate integers] (*Oeuvres,* V. II, p. 41)

*Minkowski,* Sur la réduction des formes quadratiques positives quaternaires [On the reduction of the quaternary positive quadratic forms] (*Comptes Rendus des séances de l'Académie de Paris,* V. 96, p. 1205)

The form $\sum\sum a'_{ij}x_ix_j$ obtained is said to be reduced with respect to the consecutive minima.

The binary, ternary, and quaternary positive quadratic forms can be reduced with respect to the consecutive minima. ‡ The algorithm which one uses in doing the reduction of these forms is founded on the following theorem.

For a positive quadratic form

$$f(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij}x_ix_j \quad (n = 2, 3, 4)$$

to be reduced with respect to the consecutive minima, it is necessary and sufficient that one has the inequalities

$$f(x_1, x_2, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n) \geq a_{kk} \quad (k = 1, 2, \ldots, n) \tag{2}$$

and

$$a_{11} \leq a_{22} \leq \ldots \leq a_{nn} \tag{3}$$

which is valid for integers of the variables

$$x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \quad (k = 1, 2, \ldots, n)$$

By letting

$$x_i = x'_i + \delta_i x'_k \quad \text{where } \delta_k = 0 \text{ and } i = 1, 2, \ldots, n \tag{4}$$

one will determine for the given form $f(x_1, x_2, \ldots, x_n)$ integers $\delta_1, \ldots, \delta_{k-1}, \delta_{k+1}, \ldots, \delta_n$ the condition of which the corresponding value $f(\delta_1, \ldots, \delta_{k-1}, 1, \delta_{k+1}, \ldots, \delta_n)$ would be smallest. By making successively $k = 1, 2, \ldots, n$ and repeating the procedure stated, one will always transform the given form with the aid of the substitution (4) into a form which is no different from the reduced form except by a permutation of the coefficients $(n = 2, 3, 4)$

The procedure stated in the general case can not be carried on indefinitely and one will always arrive at an equivalent quadratic form $\sum a'_{ij}x_ix_j$ which verifies the inequalities (2) and (3), but one does not know from the number of variables $n > 4$ whether the coefficients $a'_{kk}$ $(k = 1, 2, \ldots, n)$ in the form obtained exhibit a system of consecutive minima, besides: one also does not know whether the reduction of every positive quadratic form with respect to the consecutive minima is possible.

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‡ *Korkine* and *Zolotareff,* Sur les formes quadratiques positives. (*Mathematische Annalen,* V. 6, p. 336 and V. 11, p. 242)

One rids oneself of the described difficulty by changing the notation of system with $n$ consecutive minima into nothing more than considering the systems (1) which verify the equation

$$\omega = \pm 1 .$$

This is the method known as Hermite method † which has recently been improved by Mr. Minkowski in the memoir titled *Diskontinuitätsbereich für arithmetische Aquivalenz. [Discontinuity domain for arithmetical equivalence]* ‡ in the set $E$, the quadratique $\sum\sum a_{ij}x_ix_j$ being positive and $\alpha_1, \alpha_2, , \alpha_n$ any arbitrary parameters.

In the case $n = 2$ , the problem put forward has been solved by Lejeune-Dirichlet and by Hermite §

By reflecting upon the principles which have served as basis in these researches of these two illustrious geometers, I have observed that the problem introduced is intimately connected to the problem of the reduction of positive quadratic form.

In effect, Lejeune-Dirichlet and Hermite have demonstrated the following theorem.

*The conditions necessary and sufficient for which the inequality*

$$ax^2 + 2bxy + cy^2 + 2\alpha y + 2\beta y \geq 0$$

*holds, for any integral values of $x$ and $y$, in general come down to six inequalities*

$$\begin{cases} al^2 + 2blm + cm^2 \pm 2(\alpha l + \beta m) \geq 0, \\ al'^2 + 2bl'm' + cm'^2 \pm 2(\alpha l' + \beta m') \geq 0, \\ al''^2 + 2bl''m'' + cm''^2 \pm 2(\alpha l'' + \beta m'') \geq 0, \end{cases} \tag{5}$$

*where the systems of integers*

$$(l, m), (l', m') \text{ and } (l'', m'')$$

*depend only on coefficients of the quadratic form $(a, b, c)$.*

By considering the parameters $\alpha$ and $\beta$ as the Cartesian coordinates of a point $(\alpha, \beta)$ of the plane, one will determine by the inequalities (5) a hexagonal $P$ which is formed by three pairs of parallel edges.

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† *Hermite*, Extraits de lettres a Jacobi sur différents objet de la théorie des nombres (This Journal, V. 40, p. 302)

‡ This Journal, V. 129, p.220

§ *Lejeune-Dirichlet*, Mémoire cited

*Hermite,* Sur la théorie des formes quadratique ternaires [On the theory of ternary quadratic forms] (This Journal, V. 40, p. 178)

The study of properties of the hexagon $P$ plays an important role in the study of Lejeune-Dirichlet which has indicated two fundamental properties of the hexagon $P$.

*I. There exists a group of translations of the hexagon $P$ with the aid of which all the plane will be covered by the congruent hexagons.*

*II. Any binary positive quadratic form can be transformed by an equivalent form $(a, b, c)$ satisfying the conditions*

$$a - b \geq 0, b \geq 0, c - b \geq 0. \tag{6}$$

*The hexagon $P$ corresponding to the form $(a, b, c)$, in the case*

$$a - b > 0, b > 0, c - b > 0,$$

*is characterised by the systems*

$$(1, 0), (0, 1), (1, -1) \tag{7}$$

*In the case $a - b = 0$, or $(b = 0)$, or $c - b = 0$, the hexagon $P$ reduces itself into a parallelogram.*

The inequalities (6) define a domain $D$ of binary quadratic forms which is perfectly determined by the systems (7).

With the help of the substitution

$$x = x', \; y = -y',$$

one will transform the domain $D$ by a domain $D'$ defined by the inequalities

$$a + b \geq 0, -b \geq 0, c + b \geq 0 \tag{8}$$

which is characterised by the systems

$$(1, 0), (0, 1), (1, 1)$$

One calls reduced by Selling['s method] the binary positive quadratic forms which verify the inequalities (8). ‡

By effecting all the transformations of the domain $D$ with the help of substitutions

$$x = px' + qy', y = p'x' + q'y'$$

of integral coefficients and of determinant ±1, one obtains a set $(D)$ of domains of binary quadratic forms.

The set $(D)$ of domains uniformly partitions the set of all the binary positive quadratic forms, that is to say: a form which is interior to any one domain $D$ of the set $(D)$ does not belong to any other domain of this set; a form which is interior to a face of the domain $D$ belongs to only one other domain of the set $(D)$ which is contiguous to the domain $(D)$ by this face.

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‡ *Selling,* Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p.143)

The results summarised have brought me to a new point of view on the problem of reduction of positive quadratic forms.

*The problem of reduction of positive quadratic forms consist of a uniform partition of the set of positive quadratic forms with the help of domains of forms, determined using linear inequalities and enjoying the property that any substitution of integral coefficients and of determinant* $\pm 1$ *does not change the set* $(D)$ *of these domains. By partitioning the set* $(D)$ *into classes of equivalent domains and by choosing the representatives of all the classes*

$$D, D_1, \ldots, D_{m-1}, \tag{9}$$

*one will call reduced the quadratic forms which belong to these domains.*

One could attach the supplementary condition to the domains (9) by demanding: 1). that $m = 1, 2$, 2). that the positive quadratic forms interior to the domain $D$ are not equivalent and lastly, 3). that the number of linear inequalities which define the domain $D$ be the smallest one possible.

I hope to return another time to the problem posed of the reduction of positive quadratic forms.

In this mémoire, I restrict myself to the study of domains of quadratic forms which one obtains by generalising the results shown in studies of Lejeune-Dirichlet and of Hermite for the positive quadratic forms in any number of variables.

The hexagon of Lejeune-Dirichlet can be replaced for the positive quadratic forms of $n$ variables by a convex polyhedron of the analytical space in $n$ dimensions.

For a positive quadratic form $\sum\sum a_{ij}x_ix_j$, the corresponding polyhedron $R$ presents a set of points $(\alpha_i)$ verifying the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq 0 \tag{10}$$

in the set $E$. The polyhedron $R$ can be determined with the help of independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} \geq 0, \; (k = 1, 2, \ldots, \tau)$$

the number $2\tau$ of which does not exceed a limit

$$2\tau \leq 2(2^n - 1).$$

the systems of integers

$$\pm(l_{11}, l_{21}, \ldots, l_{n1}), \pm(l_{12}, l_{22}, \ldots, l_{n2}), \ldots, \pm(l_{1\tau}, l_{2\tau}, \ldots, l_{n\tau}) \qquad (11)$$

define by the corresponding equations

$$\sum\sum a_{ij} l_{ik} l_{jk} \pm 2 \sum \alpha_i l_{ik} = 0$$

$2\tau$ faces in $n-1$ dimensions of the polyhedron $R$. As these faces partition themselves into $\tau$ pairs of parallel faces, I call parallelohedron the polyhedron $R$ corresponding to any positive quadratic form.

The systems (11) enjoy many important properties.

*1. For a system* $(l_1, l_2, \ldots, l_n)$ *to belong to the series (11), it is necessary and sufficient that two systems* $(l_1, l_2, \ldots, l_n)$ *and* $(-l_1, -l_2, \ldots, -l_n)$ *are the only representations of the minimum of the form* $\sum\sum a_{ij} x_i x_j$ *in the set composed of all the systems of integers which are congruent to the system* $(l_1, l_2, \ldots, l_n)$ *by relation to the modulus 2, the system* $l_1 = 0, l_2 = 0, \ldots, l_n = 0$ *being excluded.*

*2. Among the systems (11) are found all the representations of the arithmetical minimum of the positive quadratic form* $\sum\sum a_{ij} x_i x_j$.

*3. among the systems (11) are found all the systems (1) which represent n consecutive minima of the form* $a_{ij} x_i x_j$.

*4. All the determinants which one can form of any n systems belonging to the series (11) do not exceed in numerical value a limit* $n!$.

By designating by the symbol $S_\nu$ the number of faces in $\nu$ dimensions $(\nu = 0, 1, 2, \ldots, n-1)$ of a parallelohedron $R$, I have found that

$$S_\nu \leq (n + 1 - \nu)\Delta^{(n=\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

By making $\nu = 0$ in this inequality, one obtains

$$S_0 \leq (n+1)!,$$

therefore the number of vertices of a parallelohedron $R$ does not exceed a limit $(n+1)!$. By making $\nu = n-1$, one obtains

$$S_{n-1} \leq 2(2^n - 1).$$

I demonstrate in this memoir that there exist parallelohedra, the symbol $S_\nu$ for which are expressed by the formula

$$S_\nu = (n + 1 - \nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

All these parallelohedra are primitive.

The notation of positive parallelohedra plays an important role in my studies.

I have arrived at the notation of primitive parallelohedra by observing that the parallelohedra possess Property I of hexagons of Lejeune-Dirichlet, in knowing:

*I. There exists a group of transformations of a parallelohedron R with the help of which one uniformly fills the analytical space in n dimensions by the congruent parallelohedra.*

Designate by $(R)$ the set of parallelohedra which are defined by the inequality

$$\sum\sum a_{ij}x_i x_j + 2\sum \alpha_i x_i \geq \sum\sum a_{ij} l_i l_j + 2\sum \alpha_i l_i \,,$$

$l_1, l_2, \dots, l_n$ being arbitrary integers. Ay system $(l_I)$ of integers characterise a parallelohedron of the set $(R)$.

I demonstrate that the set $(R)$ of parallelohedra corresponding to the various systems $(l_i)$ of integers unformly fills the space in $n$ dimensions.

The corresponding group of translations of the parallelohedron $R$ defined by the inequalities (10) is composed of vectors $[\lambda_i]$ which are determined by the equalities

$$\lambda_i = -\sum_{k=1}^{n} a_{ik} l_k, \quad (i = 1, 2, \dots, n)$$

$l_1, l_2, \dots, l_n$ being arbitrary integers.

Any vertice $(\alpha_i)$ of parallelohedra of the set $(R)$ belongs to at least $n+1$ parallelohedra. I call simple a vertice $(\alpha_i)$ which belongs only to $n+1$ parallelohedra of the set $(R)$ and I establish a notion of primitive parallelohedron as follows:

*One call primitive parallelohedron, a parallelohedron the vertices of which are simple.*

All the parallelohedra which are not primitive are called nonprimitive. From this point of view, the hexagon of Lejeune-Dirichlet presents a primitive parallelohedron and each parallelogram is a nonprimitive parallelohedron in two dimensions.

Any nonprimitive parallelohedron is a boundary of primitive parallelohedra and can be considered as a case of degeneracy of primitive parallelohedra.

I divide the primitive parallelohedra into various types by characterising a type of primitive parallelohedra by a set (L) of simplexes correlative to the various vertices of parallelohedra which belong to the set $(R)$.

An identical vertice $(\alpha_i)$ is determined by $n+1$ equations

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = A. \quad (k = 0, 1, 2, \ldots, n)$$

In $n+1$ systems of integers

$$(l_{1k}, l_{2k}, \ldots, l_{nk}), \quad (k = 0, 1, 2, \ldots, n)$$

I make a simplex $L$ correspond by defining it as a set of points which are determined by the equations

$$x_i = \sum_{k=0}^{n} \vartheta_k l_{ik}, \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0.$$
$$(k = 0, 1, 2, \ldots, n, i = 1, 2, \ldots, n)$$

The set $(L)$ of simplexes which are correlative to the vertices of the set $(R)$ of primitive parallelohedra enjoys important properties.

*1. The set* $(L)$ *of simplexes uniformly partition the space of* $n$ *dimensions.*

*2. By effecting the various translations of a simplex of the set* $(L)$ *the length of vector* $[l_i]$ *which are determined by the arbitrary integers* $l_1, l_2, \ldots, l_n$*, one obtains a class of congruent simplexes which belong to the set* $(L)$*.*

*3. The number of incongruent simplexes of the set* $(L)$ *is finite.*

Property II of hexagons of Lejeune-Dirichlet for the primitive parallelohedra can be generalised as follows:

*II. All the quadratic forms which define the primitive parallelohedra belonging to the type characterised by the set* $(L)$ *of simplexes are interior to a domain of quadratic form in* $\frac{n(n+1)}{2}$ *dimensions defined by linear inequalities.*

I obtains the linear inequalities which define a domain $D$ of quadratic forms corresponding to a set $(L)$ of simplexes by examining the incongruent edges of primitive parallelohedra belonging to the type characterised by the set $(L)$ of simplexes.

An vertice $(\alpha_i)$ of primitive parallelohedra of the set $(R)$ belongs to $n+1$ edges $[\alpha_i, \alpha_{ik}]$ of these parallelohedra $(k = 0, 1, 2, \ldots, n)$.

By putting

$$\alpha_{ik} - \alpha_i = p_{ik}\rho_k, \quad (i = 1, 2, \ldots, n; k = 0, 1, 2, \ldots, n)$$

one can determine the positive parameter $\rho_k$, of such manner that the numbers $p_{1k}, p_{2k}, \ldots, p_{nk}$ are integers and do not possesses common divisor. I demonstrate that the parameter $\rho_k$ expressed by a linear function

$$\rho_k = \sum \sum p_{ij}^{(k)} a_{ij} \tag{12}$$

of coefficients of the given quadratic form $\sum \sum a_{ij} x_i x_j$, the coefficients

$$p_{ij}^{(k)} = p_{jk}^{(k)}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

being rational.

I call regulator of the edge $[\alpha_i, \alpha_{ik}]$, the function $\rho_k$ determined by the formula (12); the system $(p_{ik})$ is called characteristic of the edge

$$[\alpha_i, \alpha_{ik}]. \quad (k = 0, 1, 2, \ldots, n)$$

As the edge $[\alpha_i, \alpha_{ik}]$ is correlative to a face $P_k$ of $n-1$ dimensions of the simplex $L$ which is correlative to the vertice $(\alpha_i)$, I call the function (12) *regulator of the face* $P_k$ and the system $\pm(P_{ik})$ *characteristic of the face* $P_k$ *of the simplex* $L$ $(k = 0, 1, 2, \ldots, n)$

By designating by

$$\rho_k \text{ and } \pm (p_{ik}), \quad (k = 1, 2, \ldots, \sigma)$$

the regulators and the characteristics of all the incongruent faces in $n-1$ dimensions of the set $(L)$ of simplexes, I demonstrate the following important theorem:

*The domain of quadratic forms which is characterised by the set* $(L)$ *of simplexes is defined by the linear inequalities*

$$\rho_k = \sum \sum p_{ij}^{(k)} a_{ij} \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

All the domains of quadratic forms which I have studied in this memoir possess a remarkable property: they are simple domains, that is to say the number of independent inequalities which define them is equal to $\frac{n(n+1)}{2}$.

Another coincidence has attracted my attention for a long time: that is the relation which exists within the results shown in this memoir and those which have been obtained in my first memoir titled: “On some properties of perfected positive quadratic forms” † I have observed that the set of characteristics $\pm(p_{ik}), k = 1, 2, \ldots, \sigma$ is nothing but the set of all the representations of the minimum of a perfect quadratic form $\varphi$. Thee domain $D$ either coincides well with the domain $R$ corresponding to the perfect form $\varphi$, or presents well a group of this domain.

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† This Journal, V. 133, p. 97

Despite all my effort, I have not succeeded in discovering the tie which attaches the two problems shown and which seem to be so different, abstraction made of a remarkable formula

$$\sum\sum a_{ij}x_ix_j = \frac{1}{(n-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k(p_{1k}x_1 + p_{2k}x_2 + \cdots + p_{nk}x_n)^2$$

which supplies the expression of an arbitrary quadratic form $\sum\sum a_{ij}\cdot x_ix_j$ in function of the regulators $\rho_k$ $(k = 1, 2, \ldots, \sigma)$ which are determined by the formula (12).

In this formula $\omega_k$ $(k = 1, 2, \ldots, \sigma)$ are positive integers which depend only on corresponding faces of simplexes of the set $(L)$.

To the various types of primitive parallelohedra corresponds a set $(D)$ of domains of quadratic forms. The set $(D)$ uniformly partitions the set of all the positive quadratic forms in $n$ variables.

I show in this mémoire an algorithm, by the aid of which one can determine all the domains of forms which are contiguous to a domain of the set $(D)$ by the faces in $\frac{n(n+1)}{2} - 1$ dimensions. This algorithm comes down to a certain reconstruction of the set $(L)$ of simplexes by another set $(L')$.

The set $(D)$ of domains of forms transforms into itself by all the substitutions of integral coefficients and of determinant $\pm 1$. By dividing the set $(D)$ into classes of equivalent domains, one obtains with the aid of the algorithm shown the representatives

$$D, D_1, \ldots, D_{m-1}$$

of various classes of domains belonging to the set $(D)$.

By calling reduced the quadratic forms which belong to the domains obtained, one establishes a new method of reduction of positive quadratic forms.

I have applied the general theory shown to the study of two types of primitive parallelohedra of the space in $n$ dimensions which correspond to the principal domain of quadratic forms and to the domains which are contiguous to the principal domain by the faces in $\frac{n(n+1)}{2} - 1$ dimensions. The principal domain is defined by the inequalities

$$\sum_{k=1}^{n} a_{ik} \geq 0, \quad (i = 1, 2, \dots, n)$$

$$-a_{ij} \geq 0. \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n; i \neq j)$$

I study in detail the parallelohedra of the space in 2, 3 and 4 dimensions.

In the space in 2 dimensions, there is only one type of primitive parallelohedra, provided that one does not consider as different the equivalent types; it is the hexagon of Lejeune-Dirichlet.

The set $(D)$ of domains is composed in this case of a single class, the representative of which is the principal domain defined by the inequalities (8).

In the space in 2 dimensions, there is only one single space of primitive parallelohedra – it is the parallelogram.

In the space in 3 dimensions, there is only one single type of primitive parallelohedra – it is a polyhedron of 14 faces, 8 of which are hexagonal and 6 of which are parallelogrammatic.

The set $(D)$ of domains is composed in this case of a single class, the representative of which is the principal domain. By calling reduced a ternary positive quadratic form $ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy$ which belongs to the principal domain determined with the help of inequalities

$$a + b' + b'' \geq 0, a' + b'' + b \geq 0, a'' + b + b' \geq 0, -b \geq 0, -b' \geq 0, -b'' \geq 0,$$

one will arrive at the method of reduction of ternary positive quadratic forms due to Selling. ‡

In the space in 3 dimensions, there are 4 spaces of primitive parallelohedra, they are : 1). the parallelepiped, 2). the prism of hexagonal base, 3). the parallelegrammatic dodecahedron and 4). the dodecahedron in 4 hexagonal faces and 8 parallelogrammatic faces.

In the space of 4 dimensions, there are three types of primitive parallelohedra. The set $(D)$ of domains is composed of three classes of domains of quaternary quadratic forms.

I have determined the three representatives of these classes

$$D, D', D'',$$

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‡ *Selling,* Mémoire cited

By calling as reduced the quaternary positive quadratic form which belong to the domains $D, D', D''$, I have arrived at a modification of the methods of reduction of quaternary positive quadratic forms due to Mr. Charve. †

By virtue of this theorem, the problem of uniform partition of the space in $n$ dimensions by congruent primitive parallelohedra always comes down to the study of parallelohedra corresponding to the positive quadratic forms.

I am inclined to think, without being able to demonstrate, that the theorem introduced is also true for the nonprimitive parallelohedra.

The parallelohedra of the space in 2 and in 3 dimensions have been studied by Mr. Fedorow ¶ which has discovered with the help of purely geometrical considerations, the exitence of two spaces of parallelohedra in the space in 2 dimensions and the existence of five spaces of parallelohedra in the space in 3 dimensions. Mr. Fedorow has demonstrated that there is no other parallelohedra in the space of 2 and of 3 dimensions.

The parallelohedra in 3 dimensions of Mr. Fedorow play an important role in the theory of the structure of crystals. §

# First part

## Uniform partition of the analytical space in $n$ dimensions with the aid of translations of the same convex polyhedron

## Section I

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† *Charve,* De la réduction des formes quadratiques quaternaires positives [Of the reduction of quaternary quadratic forms] (Comptes-Rendus des séances de l'Academie du Paris), V. 92, p.782 and Annales de l'École Normale supérieure, $2^{nd}$ serie, V. XI, p.119

¶ *Fedorow,* Basic principles in the theory of diagrams. St. Petersbourg, 1885 (in Russian)

*Fedorow,* Reguläre Plan- und Raumteilung. [Regular planar and space partition ] (Abhandlungen der K. bayer. Akademie der Wiss. II Cl., XX Bd. II Abt. München, 1899)

See also: *Minkowski,* Allgemeine Lehrsätze über die convexen Polyeder. [General theorems on the onvex polyhedron] (Nachrichten von der Köigl. Gesellschaft der Wissenschaften zu Göttingen, Matheem. -Physikalische Klasses, 1897, p.198)

§ See: *Fedorow,* Courses in Crystallography. St. Petersbourg, 1901 (in Russian)

*Soret,* Cristallographie physique. [Physical crystallography] Genève, 1894.

*Schönflies,* Kristallsysteme und Kristallstruktur. [Crystal systems and crystal structure] Leipzig, 1891

*Sommerfeldt,* Physikalische Kristallographie.[Physical crystallography] Leipzig, 1907.

## General properties of parallelohedra

### *On the convex polyhedra in n dimensions*

1

One will call point of the analytical space in $n$ dimensions any systems $(x_1, x_2, \ldots, x_n)$, or simply $(x_i)$, of real values of variables $x_1, x_2, \ldots, x_n$.

Consider a system of linear inequalities

$$a_{0k} + \sum_{i=1}^{n} a_{ik}x_i \geq 0 \quad (k = 1, 2, \ldots, \sigma) \tag{1}$$

of any real coefficients.

One will say that the set $R$ of points verifying the inequalities (1) is of $n$ dimensions, if there exist points satisfying the conditions

$$a_{0k} + \sum a_{ik}x_i > 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call them point, interior to the set $R$.

*Fundamental principle.* † *For the set $R$ of points verifying the inequalities (1) to be of $n$ dimensions, it is necessary and sufficient that the equation*

$$\rho_0 + \sum_{k=1}^{\sigma} \rho_k(a_{0k} + \sum a_{ik}x_i) = 0$$

*does not reduce into an identity so long as all the parameters $\rho_0, \rho_1, \ldots, \rho_\sigma$ are positive or zero.*

*Definition I. One will call convex polyhedron any set of points verifying a system of linear inequalities, on condition that this set be bounded and of $n$ dimensions.*

2

Let us suppose that the inequalities (1) define a convex polyhedron $R$ and suppose that all the inequalities (1) be independent. In such case, the polyhedron $R$ possesses $\sigma$ faces in $n-1$ dimensions which are defined by the corresponding equations

$$a_{0k} + \sum a_{ik}x_i = 0. \quad (k = 1, 2, \ldots, \sigma)$$

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† The principle announced differs only in the formulation from the fundamental principle explained in my first memoir titled: On some properties of perfect positive quadratic forms. (This journal, V. 133, p. 113)

*Definition II. Suppose that a point* $(\alpha_i)$ *belonging to* $R$ *verifies the equations*

$$a_{0r} + \sum a_{ir} x_i = 0, \quad (r = 1, 2, \ldots, \mu) \tag{2}$$

*and that one had the inequalities*

$$a_{0k} + \sum a_{ik} x_i > 0. \quad (k = \mu + 1, \ldots, \sigma)$$

*Designate by* $\nu$ *the number of dimensions of the set* $P(\nu)$ *composed of points belonging to* $R$ *and verifying the equations (2). One will call* face in $\nu$ dimensions of the polyhedron $R$ *the set* $P(\nu)a, (\nu = 0, 1, 2, \ldots, n-1)$.

In the case $\nu = 1$, one will call edge of the polyhedron $R$ a face $P(1)$ and in the case $\nu = 0$, one will call vertex of the polyhedron a face $P(0)$.

For more generality in the notations, one will designate by the symbol $P(n)$ the polyhedron $R$ itself.

Under this restriction, one can introduce the following proposition:

*Any point belonging to the polyhedron* $R$ *is interior to a face* $P(\nu)$ *of that polyhedron, where* $\nu = 0, 1, 2, \ldots, n$.

3

Let us suppose that the polyhedron $R$ possesses $s$ vertices

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is}).$$

Designate by

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{im})$$

all the vertices of $R$ which verify the equations (2).

*Theorem.* † *The face* $P(\nu)$ *in* $\nu$ *dimensions* $(\nu = 0, 1, 2, \ldots, n)$ *of the polyhedron* $R$ *defined by the equations (2) presents a set of points determined by the aid of equalities*

$$x_i = \sum_{r=1}^{m} \vartheta_r \alpha_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r \geq 0. \quad (r = 1, 2, \ldots, m)$$

*Set of domains in* $n$ *dimensions corresponding to the different vertices of a convex polyhedron.*

4

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† See my mémoire cited, Number 12

Let us suppose a vertice $(\alpha_i)$ of the polyhedron $R$ be determined by the equations

$$a_{0k} + \sum a_{ik}x_i = 0. \quad (k = 1, 2, \ldots, \mu) \tag{1}$$

*Definition. One will call domain corresponding to the vertice $(x_i)$ the set $A$ of points determined with the help of equalities*

$$x_i = \sum_{k=1}^{\mu} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, \mu) \tag{2}$$

Designate by

$$A_1, A_2, \ldots, A_s \tag{3}$$

the domains corresponding to the different vertices

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

of the polyhedron $R$. By virtue of the definition established, the set (3) of domains enjoys the following properties:

*I. All the domains of the set (3) are in $n$ dimensions.*

Let us suppose the domain $A$ determined by the equalities (2) be not in $n$ dimensions.

All the points $(x_i)$ belonging to the domain $A$ verify at least one linear equation

$$\sum p_i x_i = 0.$$

By virtue of (2), one will have

$$\sum p_i a_{ik} = 0. \ (k = 1, 2, \ldots, \mu) \tag{4}$$

As the equations (1) define a vertice $(\alpha_i)$ of the polyhedron $R$, one will find among the systems

$$(a_{11}, \ldots, a_{n1}), (a_{12}, \ldots, a_{n2}), \ldots, (a_{1\mu}, \ldots, a_{n\mu})$$

$n$ systems the determinant of which is not zero; it follow that the equalities (4) are impossible.

*II. Any point of the space in $n$ dimensions belong to at least one domain of the set (3).*

Let $(a_i)$ be an arbitrary point. Examine the sum

$$\sum a_i \alpha_{ik}, \quad (k = 1, 2, \ldots, s)$$

and suppose that the smallest sum $\sum a_i x_i$ correspond to the vertice $(\alpha_i)$ defined by the equations (1). One will have the inequalities

$$\sum a_i\alpha_{ik} \geq \sum a_i\alpha_i. \quad (k = 1, 2, \ldots, s)$$

By virtue of the theorem of Number 3, one obtains

$$\sum a_i x_i \geq \sum a_i\alpha_i,$$

for any point $(x_i)$ belonging to the polyhedron $R$.

One concludes that the inequalities

$$\sum a_i\alpha_i - \sum a_i x \geq 0 \text{ and } a_{0k} + \sum a_{ik}x_i \geq 0 \ (k = 1, 2, \ldots, \sigma)$$

can not define a polyhedra in $n$ dimensions and, by virtue of the fundamental principle of Number 1, one will have an identity

$$\rho_0 + \rho(\sum a_i\alpha_i - \sum a_i x_i) + \sum_{k=1}^{\sigma} \rho_k(a_{0k} + \sum a_{ik}x_i) = 0,$$

where

$$\rho_0 \geq 0, \rho \geq 0, \rho_k \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

By making $x_i = \alpha_i$ in this identity, it will become

$$\rho_0 + \sum_{k=1}^{\sigma} \rho_k(a_{0k} + \sum a_{ik}\alpha_i) = 0,$$

and as according to the supposition made

$$a_{0k} + \sum a_{ik}\alpha_i > 0$$

as long as $k = \mu + 1, \ldots, \sigma$, it is necessary that

$$\rho_0 = 0, \rho_{\mu+1} = 0, \ldots, \rho_\sigma = 0,$$

therefore

$$\rho(\sum a_i\alpha_i - \sum a_i x_i) + \sum_{k=1}^{\mu} \rho_k(a_{0k} + \sum a_{ik}x_i) = 0.$$

One draws

$$a_i = \sum_{k=1}^{\mu} \frac{\rho_k}{\rho} a_{ik} \text{ where } \frac{\rho_k}{\rho} \geq 0, \ (k = 1, 2, \ldots, \mu)$$

therefore the points $(\alpha_i)$ belongs to the domain $A$.

*III. A point which is interior to a face $A(\nu)$ $(\nu = 0, 1, 2, \ldots, n)$ of any domain of the set (3) belongs only to the domains of the set (3) which are contiguous by the face $A(\nu)$.*

Suppose the point $(a_i)$ be interior to a face $A(\nu)$ of the domain $A$.

By designating with

$$(a_{i1}), (a_{i2}), \ldots, (a_{i\tau}), \tau \leq \mu$$

the points which characterise the face $A(\nu)$, one can put [†]

$$a_i = \sum_{k=1}^{\tau} \rho_k a_{ik} \text{ where } \rho_k > 0. \quad (k = 1, 2, \ldots, \tau)$$

Suppose that the point $(a_i)$ be interior to another face $A'(\nu')$ of a domain $A'$ which corresponds to a vertice $(\alpha'_i)$. One can put

$$a_i = \sum_{h=1}^{\tau'} \rho'_h a'_{ih} \text{ where } \rho'_h > 0. \; (h = 1, 2, \ldots, \tau')$$

By virtue of these equalities, one will have an identity

$$\rho_0 + \sum_{h=1}^{\tau} \rho_k (a_{0k} + \sum a_{ik} x_i) = \sum_{h=1}^{\tau'} (a'_{0h} + \sum a'_{ih} x_i) \qquad (5)$$

By making within this identity $x_i = \alpha_i$, one obtains

$$\rho_0 = \sum_{h=1}^{\tau'} (a'_{0h} + \sum a'_{ih} \alpha_i),$$

and it results that

$$\rho_0 \geq 0.$$

By making within the identity (5) $x_i = x'_i$, one obtains

$$\rho_0 + \sum_{k=1}^{\tau} \rho_k (a_{0k} + \sum a_{ik} \alpha'_i) = 0;$$

consequently $\rho_0 = 0$ and

$$a_{0k} + \sum a_{ik} \alpha'_i = 0. \quad (k = 1, 2, \ldots, \tau)$$

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[†] See my mémoire cited, Number 13

In the same manner, one finds

$$a'_{0h} + \sum a'_{ih}\alpha_i = 0. \quad (h = 1, 2, \dots, \tau')$$

One concludes that the two faces $A(\nu)$ and $A'(\nu')$ coincide. ‡

By virtue of properties demonstrated of the set (3) of domains, one will say that this set uniformly *partitions* the space in $n$ dimensions.

*Definition of the group of vectors*

5

*Definition I. One will call vector the set of points determined with the help of equalities*

$$x_i = x_i + u(\alpha'_i - \alpha_i) \text{ where } 9 \leq u \leq 1, \qquad (1)$$

$(\alpha_i)$ *and* $(\alpha'_i)$ *being any two different points.*

One will designate the vector determined with the help of equalities (1) by the symbol $[\alpha_i, \alpha'_i]$. In this case $\alpha_i = 0\,(i = 1, 2, \dots, n)$, one will designate the corresponding vector by the symbol $[\alpha'_i]$ and one will call it rector of the point $(\alpha'_i)$.

*Definition II. Suppose that*

$$[\lambda_{i1}], [\lambda_{i2}], \dots, [\lambda_{im}] \qquad (2)$$

*be the vectors of arbitrary points* $(\lambda_{i1}), (\lambda_{i2}), \dots, (\lambda_{im})$. *One will call* group of vectors *the set* $G$ *of vectors determined with the help of equalities*

$$\lambda_i = \sum_{k=1}^{m} l_k \lambda_{ik},$$

$l_1, l_2, \dots, l_m$ *being of arbitrary integers.*

One will call *basis of the group* $G$ *of vectors* the vectors (2).

*Translation of polyhedra.*

6

*Definition. Effect a linear transformation of a polyhedron* $R$ *with the help of a substitution*

$$x_i = x'_i - \lambda_i, \; (i = 1, 2, \dots, n) \qquad (1)$$

*the coefficients* $\lambda_1, \lambda_2, \dots, \lambda_n$ *being arbitrary. One will say that one has effected a translation of the polyhedron* $R$ *the length of the vector* $[\lambda_i]$.

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‡ See my mémoire cited, Number 20, p. 133

Suppose that the polyhedron $R$ be determined by the inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

The transformed polyhedron $R'$ will be determined, by virtue of (1), by the inequalities

$$a_{0k} + \sum a_{ik}(x_i - \lambda_i) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call congruent the polyhedra $R$ and $R'$.

7

Let $G$ be a group of vectors. By effecting the different translations of the poly hedron $R$ the length of vectors belonging to the group $G$, one will form a set $R$ of congruent polyhedra.

One will say that the set $(R)$ of congruent polyhedra uniformly partition the space in $n$ dimensions in the following conditions.

*I. Any point of the space in $n$ dimensions belongs to at least one polyhedron of the set $(R)$.*

*II. A point which is interior to any one face $P(\nu)$ $(\nu = 0, 1, 2, \ldots, n)$ of a polyhedron of the set $(R)$ belongs to only the polyhedrons of the set $(R)$ which are contiguous by the face $P(\nu)$.*

### *Definition of parallelohedra*

8

*Definition. One will call parallelohedron any convex polyhedron $R$ possessing a group $G$ of translations with the aid of which one can uniformly fill the space in $n$ dimensions by the polyhedra congruent to the polyhedron $R$.*

By virtue of the definition established, the parallelohedra possess an important property, in knowing:

*By effecting a linear transformation of a parallelohedron with the help of a substitution by any real coefficients*

$$x_i = \alpha_{i0} + \sum_{k=1}^{n} \alpha_{ik} x'_k, \quad (i = 1, 2, \ldots, n)$$

*one obtains a convex polyhedron which is also a parallelohedron.*

Observe that by virtue of the definition established, any parallelohedron of the space in $n$ dimensions is a parallelohedron.

### *Properties of the group of vectors of a parallelohedron.*

9

Suppose that a parallelohedron $R$ be defined by the inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Designate by $G$ the group of the parallelohedron $R$ and suppose that the group $G$ possesses the basis

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{im}]. \tag{2}$$

All the vectors which form the basis of the group $G$ can not verify the same linear equation

$$\sum p_i \lambda_i = 0,$$

because otherwise the set $(R)$ of congruent parallelohedra corresponding to the group $G$ would not fill the space in $n$ dimensions.

One concludes that among the vectors (2) there are $n$ vectors

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{im}] \tag{3}$$

the determinant $\pm\Delta$ of which is not zero; one will call them independent.

*Theorem I. The numerical value $\Delta$ of the determinant of $n$ independent vectors possesses a limit*

$$\Delta \geq \int_{(R)} dx_1 dx_2 \cdots dx_n \,.$$

Let $(\alpha_i)$ be any point which is interior to the parallelohedron $R$. Introduce within our researches a parallelepiped $K$ determined with the aid of equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik}, \quad (i = 1, 2, \ldots, n) \tag{4}$$

where

$$-\delta \leq u_k \leq \delta. \quad (k = 1, 2, \ldots, n) \tag{5}$$

One can choose the positive parameter $\delta$ in such manner that all the points of the parallelohedron $R$ defined by the inequalities (1) belong to the parallelepiped $K$.

Take a positive integer $m$ and determine $(m+1)^n$ systems $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequalities

$$0 \leq l_k \leq m. \quad (k = 1, 2, \ldots, n) \tag{6}$$

Designate by

$$\lambda_i^{(h)} = \sum_{k=1}^{n} l_k^{(h)} \lambda_{ik}, \quad (h = 1, 2, \ldots, (m+1)^n) \tag{7}$$

$(m+1)^n$ corresponding vectors belonging to the group $G$.

By applying the translations of the parallelohedra $R$ the length of vectors (7), one obtains $(m+1)^n$ different parallelohedra of the set $(R)$:

$$R^{(h)}. \quad (h = 1, 2, \ldots, (m+1)^n) \tag{8}$$

Designate by $H$ a parallelepiped which is determined by the equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik}, \ (i = 1, 2, \ldots, n) \tag{9}$$

where

$$-\delta \leq u_k \leq m + \delta. \quad (k = 1, 2, \ldots, n) \tag{10}$$

I argue that all the points of parallelohedron (8) belong to the parallelepiped $H$. In effect, let $(x_i^{(h)})$ be any point of the parallelohedron $R^{(h)}$ $(h = 1, 2, \ldots, (m+1)^n)$. By posing

$$x_i = x_i^{(h)} - \lambda_i^{(h)}, \quad (i = 1, 2, \ldots, n) \tag{11}$$

one obtains a point $(x_i)$ belonging to the parallelohedron $R$ which is congruent to the point given $(x_i^{(h)})$. By virtue of (4), (7) and (11), one obtains

$$x_i^{(h)} = \alpha_i + \sum_{k=1}^{n} (l_k + u_k) \lambda_{ik},$$

and by (5) and (6), it becomes

$$-\delta \leq l_k + u_k \leq m + \delta, \quad (k = 1, 2, \ldots, n)$$

thus the point $(x_i^{(h)})$ belongs to the parallelepiped $H$.

It follows that

$$\int_{(H)} dx_1 dx_2 \cdots dx_n \geq \sum_h \int_{(R^h)} dx_1 dx_2 \cdots dx_n. \ (h = 1, 2, \ldots, (m+1)^n)$$

By observing that

$$\int_{(H)} dx_1 dx_2 \cdots dx_n = \Delta(m + 2\delta)^n$$

and that

$$\int_{(R^h)} dx_1 dx_2 \cdots dx_n = \int_{(R)} dx_1 dx_2 \cdots dx_n, \ (h = 1, 2, \ldots, (m+1)^n)$$

one obtains

$$\Delta(m + 2\delta)^n \geq (m+1)^n \int_{(R)} dx_1 dx_2 \cdots dx_n .$$

By making the number $m$ increase indefinitely, one finds

$$\Delta \geq \int_{(R)} dx_1 dx_2 \cdots dx_n .$$

*Theorem II. The group $G$ of vectors of a parallelohedron possesses basis formed by $n$ independent vectors.*

Designate by $G'$ a group of vectors having the basis (3). It can be that the two groups $G$ and $G'$ coincide. In that case $n$ vectors (3) present a basis of the group $G$.

By supposing the contrary, one will have among the vectors (2) at least one vector $[\lambda'_i]$ which does not belong to the group $G'$. By putting

$$\lambda'_i = \sum_{k=1}^{n} l'_k \lambda_{ik} ,$$

one will have among the numbers $l'_1, l'_2, \ldots, l'_n$ at least one number which is fractional.

Designate by $l_1, l_2, \ldots, l_n$ the integers verifying the inequalities

$$|l'_k - l_k| \leq \frac{1}{2} \quad (k = 1, 2, \ldots, n)$$

and suppose that $l'_r - l_r \neq 0$.

By designating

$$\lambda'_{ik} = \lambda_{ik} , \ (k = 1, 2, \ldots, n, k \neq r) \ \text{and} \ \lambda'_{ir} = \lambda'_i - \sum l_k \lambda_{ik}, ,$$

one obtains a system of $n$ independent vectors

$$[\lambda'_{i1}], [\lambda'_{i2}], \ldots, [\lambda'_{in}]$$

belonging to the group $G$, the determinant $\pm\Delta'$ of which verifies the inequality

$$0 < \Delta' \leq \frac{1}{2}\Delta .$$

The procedure explained can not be prolonged indefinitely, by virtue of Theorem I, therefore one will always obtain a system of $n$ vectors forming the basis of the group $G$.

11

*Theorem III. The numerical value $\Delta$ of the determinant of a system of $n$ vectors forming the basis of the group $G$ is expressed by the formula*

$$\Delta = \int_{(R)} dx_1 \cdot dx_2 \cdots dx_n\,.$$

suppose that the system (3) of $n$ vectors presents a basis of the group $G$.

Introduce in okur studies a parallelepiped $H'$ determined with the help of equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik},\ (i = 1, 2, \ldots, n) \tag{12}$$

where

$$\delta \le u_k \le m - \delta\,.\, (k = 1, 2, \ldots, n) \tag{13}$$

I argue that any point of the parallelepiped $H'$ belongs to at least one of parallelohedron (8). In effect, let $(x'_i)$ be any point of the parallelepiped $H'$.

Designate by $R^0$ a parallelohedron of the set $(R)$ to which belongs the point $(x'_i)$. Let $[\lambda_i]$ be the vector which defines a translation of the parallelohedron $R$ to $R^0$. By putting

$$x_i = x'_i - \lambda_i\,, \tag{14}$$

one obtains a point $(x_i)$ belonging to the parallelohedron $R$ which is congruent to the point $x'_i$. By virtue of the supposition made, the vector $[\lambda_i]$ can be determined by the equalities

$$\lambda_i = \sum_{k=1}^{n} l_k \lambda_{ik}\,. \tag{15}$$

As the point $(x'_i)$ belongs to the parallelepiped $H'$, one will present the equalities (14), by (12) and (15), in the following form:

$$x + i = \alpha_i + \sum_{k=1}^{n} (u_k - l_k) \lambda_{ik}\,.$$

The point $(x_i)$ belonging to the parallelohedron $R$ belongs also, by virtue of the supposition made, to the parallelepiped $K$ determined by the equalities (4), by condition of (5). It follows that

$$-\delta \leq u_k - l_k \leq \delta\,, \quad (k = 1, 2, \ldots, n)$$

and as, by (13),

$$\delta \leq u_k \leq m - \delta\,, \quad (k = 1, 2, \ldots, n)$$

it becomes

$$0 \leq l_k \leq m, \quad (k = 1, 2, \ldots, n)$$

therefore the vector $[\lambda_i]$ determined by the equalities (5) is among the vectors (7) and the point examined $(x')$ of the parallelepiped $H'$ belongs to a parallelohedron of the series (8).

It follow that

$$\int_{H'} dx_1 dx_2 \cdots dx_n \leq \sum_h \int_{(R^h)} dx_1 dx_2 \cdots dx_n\,.$$

By making the number $m$ grow indefinitely, one obtains

$$\Delta \leq \int_{(R)} dx_1 dx_2 \cdots dx_n\,.$$

By virtue of Theorem I, it is necessary that

$$\Delta = \int_{(R)} dx_1 dx_2 \cdots dx_n\,.$$

*Properties of faces in $n-1$ dimensions of a parallelohedron.*

12

Suppose that a parallelohedron $R$ be defined by the independent inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0\,. \quad (k = 1, 2, \ldots, \delta)$$

Designate by $P_k (k = 1, 2, \ldots, \sigma)$ the faces in $n-1$ dimensions of the parallelohedron $R$ determined by the corresponding equations

$$a_{0k} + \sum a_{ik} x_i = 0\,. \tag{1}$$

Let $(\alpha_i)$ be a point which is interior to the face $P_k$. Examine a parallelepiped $K$ defined by the equalities

$$x_i = \alpha_i + u_i \;\text{ where }\; |u_i| \leq \epsilon\,. (i = 1, 2, \ldots, n) \tag{2}$$

One can choose a parameter $\epsilon$ however small that one will have the inequalities

$$a_{0r} + \sum a_{ir} x_i > 0, \ (r = 1, 2, \ldots, \sigma, r \neq k) \tag{3}$$

for any point $(x_i)$ of the parallelepiped $K$. It results in that all the points of the parallelepiped $K$ verifying the inequality

$$a_{0k} + \sum a_{ik} x_i \geq 0 \tag{4}$$

belong to the parallelohedron $R$. As the point $(\alpha_i)$ verifies the equation (1), the equation (4) reduces, by reason of (2), to this one here

$$\sum a_{0k} u_i \geq 0 \,.$$

I argue that one can choose a value of the parameter $\epsilon$ however sall that all the points of the parallelepiped $K$ verifying the inequality

$$\sum a_{ik} u_i \leq 0$$

will belong to another parallelohedron $R_k$ of the set $(R)$. By relying on the demonstrated properties of the group $G$ of vectors, one will easily demonstrate the proposition stated.

Two parallelohedra $R$ and $R_k$ are contiguous by the face $P_k$ in $n-1$ dimensions. Designate by $[\lambda_{ik}]$ the vector which defines a translation of the parallelohedron $R_k$ to $R$. The face $P_k$ which is defined in the parallelohedron $R$ by the equation (1) will be defined in the parallelohedron $R_k$ by the equation

$$-a_{0k} - \sum a_{ik} x_i = 0 \,. \tag{5}$$

By carrying out a translation of the face $P_k$ the length of the vector $\lambda_{ik}$, one obtains another face $P_k$ of the parallelohedron $R$ which will be within the parallelohedron determined by the equation

$$-a_{0k} - \sum a_{ik}(x_i - \lambda_{ik}) = 0 \,.$$

One will call parallel the faces $P_k$ and $P'_k$ of the parallelohedron $R$. We have arrived at the following important result:

*All the faces in $n-1$ dimensions of a parallelohedron can be divided into pairs of parallel faces.*

13 Designate by

$$R_1, R_2, \ldots, R_\sigma$$

all the parallellohedra which are contiguous to the parallelohedron $R$ by the faces $P_1, P_2, \ldots, P_\sigma$. Designate by

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{i\sigma}] \tag{6}$$

the corresponding vectors.

By virtue of the definition of the parallelohedron, the vectors (6) form the basis of the group $G$. Among the vectors of this group there exist the systems of $n$ vectors which form a basis of the group $G$.

*Congruent faces in different dimensions of a parallelohedron.*

14

Suppose that a face $P(\nu)$ in $\nu$ dimensions of a parallelohedron $R$ also belongs to the parallelohedra

$R_1, R_2, \ldots, R_\tau$ of the set $(R)$. Let $(\alpha_i)$ be a point which is interior to the face $P(\nu)$. One can determine a positive value of the parameter $\epsilon$ in such a manner that all the point of the parallelepiped $K$ defined by the equalities

$$x_i = \alpha_i + u_i \text{ where } |u_i| \leq \epsilon \quad (i = 1, 2, \ldots, n)$$

belong to the parallelohedra $R, R_1, R_2, \ldots, R_\tau$.

Designate by $[\lambda_{ik}]$ the vectors the length of which one will carry out the translations of parallelohedra $R_k$ into $R\,(k = 1, 2, \ldots, \tau)$.

By carrying out the translations of the face $P(\nu)$ the length of vectors $[\lambda_{ik}]\,(k = 1, 2, \ldots, \tau)$, one obtains the new faces

$$P'(\nu), P''(\nu), [\ldots, ]P^{(\tau)}(\nu)$$

of the parallelohedron $R$.

*Definition I. One will call congruent the faces of the parallelohedron R*

$$P'(\nu), P''(\nu), \ldots, P^{(\tau)}(\nu)$$

*in $\nu$ dimensions $(\nu = 0, 1, 2, \ldots, n-1)$.*

15

*Theorem. The number of parallelohedra of the set $(R)$ which are contiguous by the same face in $\nu$ dimensions can not be less than $n + 1 - \nu\,(\nu = 0, 1, 2, \ldots, n-1)$.*

Suppose that the face $P(\nu)$ be determined within the parallelohedron $R$ by the equation

$$a_{0r} + \sum a_{ir}x_i = 0. \; (r = 1, 2, \ldots, \mu) \tag{1}$$

Designate by $R_1, R_2, \ldots, R_\mu$ the parallelohedra which are contiguous to $R$ by the faces in $n-1$ dimensions defined by the equations (1). The face $P(\nu)$ will belong to all the parallelohedra $R_1, R_2, \ldots, R_\mu$, therefore

$$\tau \geq \mu.$$

As the face $P(\nu)$ is in $\nu$ dimensions, it is necessary that

$$\mu \geq n - \nu,$$

and as a result

$$\tau \geq n - \nu.$$

16

*Definition II. One will call simple a face in $\nu$ dimensions which belong to only $n + 1 - \nu$ parallelohedra of the set $(R)$.*

*Definition II. One will call primitive a parallelohedron, all the faces in different dimensions of which are simple.*

The primitive parallelohedra possess many important properties which simplify the study.

In the subsequent studies, one will study only the primitive parallelohedron and all the nonprimitive parallelohedra which can be considered as a boundary of primitive parallelohedra.

I am inclined to think that each primitive parallelohedron can be considered in this point of view, but I have not been successful in demonstrating this.

## Section II

### Fundamental properties of primitive parallelohedra

*Definition of primitive parallelohedra.*

17

We have called in Number 16 “primitive parallelohedron” all parallelohedron, all the faces in different dimensions of which are simple.

*Theorem I. for a parallelohedron to be primitive it is necessary and sufficient that all the vertices be simple.*

The theorem stated is evident by virtue of the definition established.

*Theorem II. Two primitive parallelohedra belonging to the set $(R)$ can be contiguous by only one face in $n-1$ dimensions.*

Suppose that a face $P(\nu)$ in $\nu$ dimensions of a primitive parallelohedron $R$ be determined with the aid of $n - \nu$ equations

$$a_{0r} + \sum a_{ir} x_i = 0. \; (r = 1, 2, \ldots, n - \nu) \qquad (1)$$

Designate by $R_1, R_2, \dots, R_{n-\nu}$ the parallelohedra which are contiguous to the parallelohedron $R$ by the faces in $n-1$ dimensions defined by the aid of equations (1). The face $P(\nu)$ will not belong to the parallelohedra $R_1, R_2, \dots, R_{n-\nu}$ by virtue of the definition established, thus the theorem introduced is demonstrated.

*Edges of primitive parallelohedra of the set* $(R)$

18 Let $(\alpha_i)$ be a vertice of the primitive parallelohedron $R$ determined by $n$ equations

$$a_{0k} + \sum a_{ik} x_i = 0. \ (k = 1, 2, \dots, n) \tag{1}$$

Designate by $R_1, R_2, \dots, R_n$ the parallelohedra contiguous to the parallelohedron $R$ by the faces in $n-1$ dimensions determined with the help of equation (1).

By virtue of the definition established, the vertice $(\alpha_i)$ will not belong to the parallelohedra $R_1, R_2, \dots, R_n$ of the set $(R)$.

Determine $n$ numbers $P_{1k}, P_{2k}, \dots, P_{nk}$ with the help of equations

$$\sum a_{ir} P_{ik} = 0. \ (r = 1, 2, \dots, n; r \neq k; k = 1, 2, \dots, n) \tag{2}$$

The equations (2) do not define the number $P_{1k}, P_{2k}, \dots, P_{nk}$ to a common factor. Attach to the equations (2) a condition

$$\sum a_{ik} p_{ik} > 0 \ (k = 1, 2, \dots, n) \tag{3}$$

and consider a vector $g_k$ determined with the help of equalities

$$x_i = \alpha_i + p_{ik}\rho \ \text{ where } \ \rho \geq 0.$$

By attributing to the parameter $\rho$ positive values sufficiently small, one will determine, by (3), the points of the vector $g_k$ belonging to $R$. By putting

$$\alpha_{ik} = \alpha_i + p_{ik}\rho_k,$$

one will determine a vertice $(\alpha_{ik})$ of the parallelohedron $R$ adjacent to the vertice $(\alpha_i)$ by an edge $P_k(1)$ of the parallelohedron $R$ $(k = 1, 2, \dots, n)$. One will characterise the edge $P_k(1)$ by the symbol $[\alpha_i, \alpha_{ik}]$.

Observe that all the points of the edge $P_k(1)$ verifies $n-1$ equations

$$a_{0r} + \sum a_{ir} x_i = 0. \quad (r = 1, 2, \dots, n; r \neq k)$$

It follows that the edge $P_k(1)$ belongs to the parallelohedra

$$R, R_1, \ldots, R_{k-1}, R_{k+1}, \ldots, R_n \quad (k = 1, 2, \ldots, n)$$

and by virtue of the definition established, does not belong to any other parallelohedron of the set $(R)$.

One concludes that the parallelohedra

$$R_1, R_2, \ldots, R_n$$

are contiguous by an edge too. By designating this edge by $P_0(1)$, one will determine it with the symbol $[\alpha_i, \alpha_{i0}]$ by putting

$$\alpha_{i0} = \alpha_i + p_{i0}\rho_0.$$

We have arrived at the following result:

*There exist* $n+1$ *edges of parallelohedra of the set* $(R)$*, contiguous by one common vertex of these parallelohedra.*

Observe that $n-1$ edges

$$P_1(1), \ldots, P_{k-1}(1), P_{k+1}(1), \ldots, P_k(1) \quad (k = 1, 2, \ldots, n)$$

define a face in $n-1$ dimensions which is common to the parallelohedra $R$ and $R_k$ $(k = 1, 2, \ldots, n)$. Two parallelohedra $R_k$ and $R_h$ $(k = 1, 2, \ldots, n; h = 1, 2, \ldots, n)$ are contiguous by a face in $n-1$ dimensions which is defined by $n-1$ edges

$$P_r(1). \quad (r = 0, 1, 2, \ldots, n; r \neq k, r \neq h)$$

*Canonical form of equations which define a vertex of a primitive parallelohedron.*

19

By conserving the previous notations, one can determine the vertice $(\alpha_i)$ within the parallelohedron $R$ with the help of equations

$$u_k(a_{0k} + \sum a_{ik}x_i) = 0, \ (k = 1, 2, \ldots, n) \tag{1}$$

$u_1, u_2, \ldots, u_n$ being positive arbitrary parameters. One will say that the equation

$$-u_k(a_{0k} + \sum a_{ik}x_i) = 0 \text{ where } u_k > 0$$

does not define within the parallelohedron $R$ a face in $n-1$ dimensions because the inequality

$$-u_k(a_{0k} + \sum a_{ik}x_i) \geq 0$$

will not satisfy all the points of the parallelohedron $R$.

*Theorem. One can determine the positive values of parameters* $u_1, u_2, \dots, u_n$ *to a common factor, such that by putting*

$$a'_{0k} = u_k a_{0k}, a'_{ik} = u_k a_{ik}, \ (i = 1, 2, \dots, n; k = 1, 2, \dots, n)$$

*one will define the vertice* $(\alpha_i)$ *within the parallelohedron* $R$ *by the equations*

$$\sum a'_{ik}(x_i - \alpha_i) = 0, \ (k = 1, 2, \dots, n) \tag{2}$$

*and one will define the vertice* $(\alpha_i)$ *within the parallelohedron* $R_k$ $(k = 1, 2, \dots, n)$ *by the equations*

$$\left\{ \begin{array}{l} \sum (a'_{ih} - a'_{ik})(x_i - \alpha_i) = 0, \ (h = 1, 2, \dots, n; h \neq k) \\ -\sum a'_{ik}(x_i - \alpha_i) = 0. \ (k = 1, 2, \dots, n) \end{array} \right. \tag{3}$$

Take an arbitrary positive parameter $\delta$ and determine the parameters $u_1, u_2, \dots, u_n$ after the equations

$$u_k \sum a_{ik}(\alpha_i - \alpha_{i0}) = \delta. \ (k = 1, 2, \dots, n) \tag{4}$$

I argue that the values $u_1, u_2, \dots, u_n$ obtained satisfy the conditions of the theory stated.

To demonstrate this, observe in the first place that the equations (4) define the positive values of $u_1, u_2, \dots, u_n$. In effect, we have seen in Number 18 that the edge $P_0(1)$ defined by the equalities

$$x_i = \alpha + u(\alpha_{i0} - \alpha_i) \text{ where } 0 \leq u \leq 1 \tag{5}$$

does not belong to the parallelohedra $R_1, R_2, \dots, R_n$. One concludes that by attributing to the parameter $u$ any negative values sufficiently small, one will determine by the equality (5) a point which will be interior to the parallelohedron $R$. It follows that

$$\sum a_{ik}(\alpha_i - \alpha_{i0}) > 0, \quad (k = 1, 2, \dots, n)$$

and the equations (4) give

$$u_k > 0. \quad (k = 1, 2, \dots, n)$$

This established, designate by

$$\sum a_{ir}^{(k)}(x_i - \alpha_i) = 0 \ (r = 1, 2, \dots, n) \tag{6}$$

the equations which define the vertice $(\alpha_i)$ in the parallelohedron $R_k$ $(k = 1, 2, \dots, n)$

Observe that $n$ edges $P_r(1)\,(r = 0, 1, 2, \dots, n, r \neq k)$ are contiguous by the vertice $(\alpha_i)$ in the parallelohedron $R_k$. Each equation (6) will be verified by $n - 1$ edges. One can thus put

$$\begin{cases} \sum a_{ik}^{(k)}(\alpha_{ir} - \alpha_i) = 0, \ (r = 1, 2, \dots, n; r \neq k) \\ \sum a_{ik}^{(k)}(\alpha_{i0} - \alpha_i) > 0 \end{cases} \tag{7}$$

and

$$\begin{cases} \alpha a_{ih}^{(k)}(\alpha_{ir} - \alpha_i) = 0, \ (r = 0, 1, 2, \dots, n; r \neq k, r \neq h) \\ \sum a_{ih}^{(k)}(\alpha_{ih} - \alpha_i) > 0. \ (h = 1, 2, \dots, n; h \neq k) \end{cases} \tag{8}$$

The conditions established define the coefficients of equations (6) to a common positive factor, which can be arbitrarily chosen.

Observe that the coefficients of equations (1), which define the vertice $(\alpha_i)$ in the parallelohedron $R$ are also determined to a common positive factor and satisfy the conditions

$$\begin{cases} \sum a_{ik}(\alpha_{ir} - \alpha_i) = 0, \ (r = 1, 2, \dots, n; r \neq k; k = 1, 2, \dots, n) \\ \sum a_{ik}(\alpha_{ik} - \alpha_i) > 0 \end{cases} \tag{9}$$

The equalities (4), (7), (8) and (9), one takes

$$\left. \begin{aligned} a_{ik}^{(k)} &= -\delta_k u_k a_{ik}, \\ a_{ih}^{(k)} &= \delta_h (u_h a_{ih} - u_k a_{ik}), \end{aligned} \right\} (i = 1, 2, \dots, n; h = 1, 2, \dots, n; h \neq k)$$

where $\delta_1, \delta_2, \dots, \delta_n$ are positive factors. One can put

$$\delta_1 = 1, \delta_2 = 1, \dots, \delta_n = 1,$$

and the equations (6) become

$$\sum (u_h a_{ih} - u_k a_{ik})(x_i - \alpha_i) = 0, \ (h = 1, 2, \dots, n; h \neq k)$$
$$-\sum u_k a_{ik}(x_i - \alpha_i) = 0.$$

The theorem introduced is thus demonstrated.

One will say that the equations (2) and (3) which define the vertice $(\alpha_i)$ in the contiguous parallelohedron $R, R_1, \dots, R_n$ are presented in the canonical form.

We have seen in Number 18 that the parallelohedra $R_k$ and $R_h\,(k = 1, 2, \dots, n; h = 1, 2, \dots, n)$ are contiguous by a face in $n - 1$ dimensions. As this face is characterised by the edges $P_r(1)\,(r =$

$0, 1, 2, \ldots, n; r \neq k; r \neq h$), one will determine it in the parallelohedron $R_h$ by the canonical equation

$$\sum (a'_{ik} - a'_{ih})(x_i - \alpha_i) = 0.$$

*Canonical form of inequalities which define a positive parallelohedron.*

20

Suppose that a primitive parallelohedron $R$ is determined with the help of independent inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Bu designating with $u_1, u_2, \ldots, u_\sigma$ of arbitrary positive parameters, one will determine the parallelohedron $R$ with the help of independent inequalities

$$u_k (a_{0k} + \sum a_{ik} x_i) \geq 0. \ (k = 1, 2, \ldots, \sigma) \tag{1}$$

We will see how all the problem of the study of primitive parallelohedra comes down to the appropriate choice of parameters $u_1, u_2, \ldots, u_\sigma$.

*Fundamental Theorem. One can determine the positive values of parameters $u_1, u_2, \ldots, u_\sigma$ to a common factor, such that by putting*

$$a'_{0k} = u_k a_{0k}, a'_{ik} - u_k a_{ik}, \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, \sigma)$$

*one will determine the parallelohedron $R$ with the help of inequalities*

$$a'_{0k} + \sum a'_{ik} x_i \geq 0 \ (k = 1, 2, \ldots, \sigma) \tag{2}$$

*which enjoy the following property: all the vertices of the parallelohedron $R$ will be determined by the equations presented in the canonical form.*

One will call the inequalities (2) canonical.

By conserving the previous notations, suppose that one had chosen the parameters $u_1, u_2, \ldots, u_n$ in such a manner that the vertice $(\alpha_i)$ is determined by the canonical equations

$$a'_{0k} + \sum a'_{ik} x_i = 0. \quad (k = 1, 2, \ldots, n)$$

Examine the equations which define a vertice $(\alpha_{ik})$ $(k = 1, 2, \ldots, n)$ of the parallelohedron $R$ adjacent to the vertice $(\alpha_i)$ by the edge $P_k(1)$.

The vertice $(\alpha_{ik})$ satisfies $n-1$ equations

$$a'_{0h}+\sum a'_{ih}x_i=0.\ (h=1,2,\ldots,n;h\neq k) \tag{3}$$

Designate by

$$b_{0k}+\sum b_{ik}x_i=0 \tag{4}$$

the $n^{th}$ equation which defines the vertice $(\alpha_{ik})$.

Determine the positive parameters $v_h\,(h=1,2,\ldots,n,h\neq k)$ and $v_k$ corresponding to the equations (3) and (4), which reduces to these equations in the canonical form:

$$v_h(a'_{0h}+\sum a'_{ih}x_i)=0\quad (h=1,2,\ldots,n;h\neq k)$$

and

$$v_k(b_{0k}+\sum b_{ik}x_i)=0.$$

I argue that one can put

$$v_h=1.\quad (h=1,2,\ldots,n;h\neq k)$$

To demonstrate this, examine the canonical equation which defines in the parallelohedron $R_h\,(h=1,2,\ldots,n;h\neq k)$ a face in $n-1$ dimensions common to the parallelohedra $R_r$ and $R_h$ $(r=1,2,\ldots,n;r\neq h,r\neq,k)$.

By virtue of the theorem of Number 19, this face will be determined within $R_h$ by the canonical equation

$$\sum(a'_{ir}-a'_{ih})(x_i-\alpha_i)=0.$$

Besides, this same face will be determined in $R_h$, by virtue of the supposition made, by the canonical equation

$$\sum(v_ra'_{ir}-v_ha'_{ih})(x_i-\alpha_i)=0.$$

It results in that

$$v_ra'_{ir}-v_ha'_{ih}=\delta(a'_{ir}-a'_{ih}),\quad (i=1,2,\ldots,n)$$

and so

$$v_r=\delta,v_h=\delta,$$

thus

$$v_r=v_h.\quad (r=1,2,\ldots,n;r\neq k;r\neq h)$$

As the parameters $v_h\,(h=1,2,\ldots,n)$ are defined to a factor, one can put

$$v_h=1,\quad (h=1,2,\ldots,n;h\neq k)$$

and it only remains to determine the parameter $v_k$ in order to define the vertice $(\alpha_{ik})$ by the canonical equations.

21

By applying the procedure explained to all the vertices of the parallelohedron $R$ adjacent to the vertice $(\alpha_i)$ and so on, one will successively determine the values of various parameters corresponding to all the inequalities (1).

It can turn out that one determines for one inequality the value of the corresponding parameter in various manners. I argue that all these values of the same parameter coincide.

The problem posed is extremely difficult. It is within this group of studies explained that is manifested their true geometrical characteristic, and one does not manage to master the difficulties which arise as a result with the help of geometrical methods.

*Set of simplexes corresponding to the various vertices of a primitive parallelohedron.*

22

We have seen in Number 4 that the various vertices of a parallelohedron $R$ $(\alpha_{i1}), (\alpha_{i2}), \dots, (\alpha_{is})$ correspond to the domains

$$A_1, A_2, \dots, A_s \tag{1}$$

which uniformly fill the space in $n$ dimensions.

By conserving the previous notations, examine a domain $A$ which corresponds to the vertice $(\alpha_i)$ of the parallelohedron $R$.

The domain $A$ is composed of points determined by the equalities

$$x_i = \sum_{k=1}^{n} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \dots, n) \tag{2}$$

The domain $A$ possesses $n$ faces in $n-1$ dimensions which correspond to $n$ edges $P_k(1), (k = 1, 2, \dots, n)$ contiguous by the vertice $(\alpha_i)$.

One will call the domain $A$ simple.

Extract from the domain $A$ a simplex $L$ by the solution with the help of equalities

$$x_i = \sum_{k=1}^{n} \vartheta_k u_k a_{ik} \text{ where } \sum \vartheta_k \leq 1 \text{ and } \vartheta_k \geq 0. \ (k = 1, 2, \dots, n)$$

The simplex $L$ possesses $n+1$ faces in $n-1$ dimensions which are opposite to $n+1$ vertices

$$(0), (u_1 a_{i1}), (u_2 a_{i2}), \ldots, (u_n a_{in}),$$

Examine the face of the simplex $L$ which is opposite to the vertex (0). One can present the equation which defines this face in the form

$$1 - \sum p_i x_i = 0. \qquad (3)$$

It follows that one will have an inequality

$$1 - \sum p_i x_i > 0$$

for any point of $L$ which does not belong to the face examined.

As the vertices of $L$: $(u_k a_{ik}) (k = 1, 2, \ldots, n)$ satisfy the equation (3), one has

$$\sum p_i a_{ik} = \frac{1}{u_k}, \ (k = 1, 2, \ldots, n) \qquad (4)$$

thus

$$\sum p_i a_{ik} > 0. \quad (k = 1, 2, \ldots, n).$$

By virtue of (2), one obtains the inequality

$$\sum p_i x_i > 0$$

which holds for any point $(x_i)$ of the domain $A$, the vertex (0) being excluded.

23

Examine in the same manner $n$ domains $A_1, A_2, \ldots, A_n$ which are contiguous to the domain $A$ by faces in $n-1$ dimensions.

One will take from the simple domain $A_k a, (k = 1, 2, \ldots, n)$ defined by the equalities

$$x_i = \sum \rho_h a_{ih} + \rho_k b_{ik} \text{ where } \rho_k \geq 0 \text{ and } \rho_h \geq 0, \ (h = 1, 2, \ldots, n; h \neq k)$$

a simplex $L_k$ composed of points

$$x_i = \sum \vartheta_h u_h a_{ih} + \vartheta_k v_k b_{ik} \text{ where } \sum \vartheta_h + \vartheta_k \leq 1, \vartheta_k \geq 0, \vartheta_h \geq 0 \quad (h = 1, 2, \ldots, n; h \neq k)$$

Designate by

$$1 - \sum p_{ik} x_i = 0$$

the equation of the face of the simplex $L_k$ which is opposite to the vertex (0).

One will have the equalities

$$\sum p_{ik} a_{ih} = \frac{1}{u_b} \quad (h = 1, 2, \ldots, n; h \neq k)$$

and

$$\sum p_{ik} b_{ih} = \frac{1}{u_k}.$$

By virtue of equalities (4), one obtains

$$\sum p_{ik} a_{ih} = \sum p_i a_{ih}. \quad (h = 1, 2, \ldots, n; h \neq k)$$

It follows that

$$\sum p_{ik} x_i = \sum p_i x_i, \tag{5}$$

for any point $(x_i)$ belonging to the face common to domains $A$ and $A_k$.

By applying the procedure explained to the domains which are contiguous to the domains $A_1, A_2, \ldots, A_n$ and so on, one will extract from any domain of the set (1) a corresponding simplex.

It can turn out that one extracts from the same domain the corresponding simplex by various manners. I argue that all these simplexes coincide.

It is clear that the problem stated does not differ from a formulation of the problem put forward in Number 21.

We shall show a new formulation of this problem.

*On a function defined by the set of simplexes corresponding to the various vertices of a primitive parallelohedron.*

24 Introduce within our study a function $P(x_1, x_2, \ldots, x_n)$ of variables $x_1, x_2, \ldots, x_n$ by defining as follows.

1. One will determine the function $P(x_1, x_2, \ldots, x_n)$ in the domain $A$ by the formula

$$P_{(A)}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} p_i x_i.$$

2. In the domains $A_k$ $(k = 1, 2, \ldots, n)$ contiguous to the domain $A$ by the faces in $n - 1$ dimensions, one will determine the function $P(x_1, x_2, \ldots, x_n)$ by the formula

$$P_{(A_k)}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} P_{ik} x_i. \quad (k = 1, 2, \ldots, n)$$

Let

$$A, A', A'', \dots, A^{(m)} \tag{1}$$

be a series of domains which are successively contiguous by the faces in $n-1$ dimensions. One will successively take from these domains the following simplexes.

$$L, L', L'', \dots, L^{(m)}$$

and one will determine the corresponding function.

$$\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i' x_i, \sum_{i=1}^{n} p_i'' x_i, \dots, \sum_{i=1}^{n} p_i^{(m)} x_i.$$

One will define the function $P(x_1, x_2, \dots, x_n)$ in the domains (1) by the formula

$$P_{(A^{(k)})}(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} p_i^k x_i. \quad (k = 1, 2, \dots, m)$$

*Fundamental Theorem. The function $P(x_1, x_2, \dots, x_n)$ defined by the conditions 1, 2, and 3 is continuous and uniform in all the space in n dimensions.*

Observe that the fundamental introduced only give as a new formulation of the fundamental theorem of Number 20.

Take an arbitrary closed contour $C$. By traversing the contour $C$, one can determine a series of domains successively contiguous by faces in $n-1$ dimensions in which belong the points of the contour $C$:

$$A^{(0)}, A', A'', \dots, A^{(m)}, A^{(0)}, A', \dots$$

To demonstrate this, take a point $(\xi_{i0})$ of the contour $C$ and designate by $C_0$ a curve which is being traversed within a domain $A^{(0)}$ leaving from the initial point $(\xi_{i0})$. Suppose that the curve $C_0$ does not coincide with the contour $C$ and designate by $(\xi_{i1})$ the final point of the curve $C_0$.

Suppose that on leaving the point $(\xi_i)$ one got out of the domain $A^{(0)}$ and that one entered inside the domain $A'$. Designate by $C_1$ a group of contour $C$ which one has traversed in the domain $A'$ when leaving the point $(\xi_{i1})$ and so on and so forth. Suppose that one had divided with the help of the procedure described the contour $C$ into groups

$$C_0, C_1, \dots, C_m, C_0$$

which belong to the domains

$$A^{(0)}, A', A'', \dots, A^{(m)}, A^{(0)}. \tag{2}$$

It can turn out that two adjacent domains of this series $A^{(k)}$ and $A^{(k+1)}$ are not contiguous by a face in $n-1$ dimensions. One inserts in this case between the domains $A^{(k)}$ and $A^{(k+1)}$ new domains of the solutions as follows:

A point $(\xi_{i,k+1})$ which is the final point of the curve $C_k$ and which gives the initial point of the curve $C_{k+1}$ belongs, by virtue of the supposition made, to the domains $A^{(k)}$ and $A^{(k+1)}$. One concludes that the point $(\xi_{i,k+1})$ is interior to a face $A^{(k)}(\nu)$ in $\nu$ dimensions which is common to the domains $A^{(k)}$ and $A^{(k+1)}$.

One can determine a parallelepiped $K$ with the help of equalities

$$x_i = \xi_{i,k+1} + u_i \text{ where } |u_i| \leq \epsilon, \ (i = 1, 2, \dots, n)$$

of manner such that the points of the parallelepiped $D$ do not belong to the domains of the series (1) (Number 22) which are contiguous by the face $A^{(k)}(\nu)$.

Take with the parallelepiped $K$ two points $(x_{ik})$ and $x_{i,k+1}$ which ae interior to the domain $A^{(k)}$ and $A^{(k+1)}$ and take within the parallelepiped $K$ a curve $C^{(k)}$ which joins the points $(x_{ik})$ and $(x_{i,k+1})$. One choose this curve in such a manner that it does not pass beyond any face of domains (1) (Number 22) of which the number of dimension $s$ is less than $n-1$.

Suppose that the curve $C^{(k)}$ traverse the domain

$$A^{(k)}, A_1^{(k)}, \dots, A_\mu^{(k)}, A^{(k+1)}.$$

By virtue of the supposition made, the domains obtained are successively contiguous by the faces in $n-1$ dimensions. All these domains are contiguous pairwisely by the face $A^{(k)}(\nu)$.

In the same manner, one will examine all the pairs of adjacent domains of the series (2) and one will form the series

$$A^{(0)}, A', \dots, A^{(m)}, A^{(0)}$$

of domains successively contiguous by the faces in $n-1$ dimensions to which belong all the points of the closed contour $C$ given.

25

This established, observe that the fundamental theorem introduced is true in the case where all the domains (2) are contiguous in at least one edge.

In effect, suppose that one had successively taken away from the domains (2) the simplexes

$$L^{(0)}L', L'', \dots, L^{(m)}, L^{(m+1)} \tag{3}$$

I argue that the simplex $L^{(m+1)}$ taken from the domain $A^{(0)}$ coincide with the simplex $L^{(0)}$. To demonstrate this, designate by

$$\sum p_i^{(m+1)} x_i = \delta \sum p_i^{(0)} x_i \,. \tag{4}$$

By virtue of the supposition made, the domains (2) are contiguous by at least one edge. Let $(a_i)$ be a point of this edge.

As the domains $A^{(0)}$ and $A'$ are contiguous by a face in $n-1$ dimensions, one will have, as we have seen this in Number 23, an equality

$$\sum p_i^{(0)} x_i = \sum p_i' x_i \tag{5}$$

which holds for any point $(x_i)$ of thee face common to the domains $A^{(0)}$ and $A'$.

By making $x_i = a_i$, one obtains

$$\sum p^{(0)} a_i = \sum p_i' a_i$$

In the same manner, one will obtain

$$\sum p_i^{(0)} a_i = \sum p_i' a_i = \dots \sum p_i^{(m)} a_i = \sum p_i^{(m+1)} a_i \,.$$

On the other hand, the identity (4) gives

$$\sum p_i^{(0)} a_i = \delta \sum p_i^{(m+1)} a_i \,,$$

and as $\sum p_i^{(0)} a_i > 0$, then $\delta = 1$, therefore

$$\sum p_i^{(m+1)} x_i = \sum p_i^{(0)} x_i$$

and the two simplexes $L^{(0)}$ and $L^{(m+1)}$ coincide.

By virtue of the definition established, one will determine the function $P(x_1, \dots, x_n)$ in the domain $A^{(0)}$ by the formula

$$p_{(A^0)}(x_1, x_2, \dots, x_n) = \sum p_i^(0) x_i$$

by leaving the domain $A^{(0)}$ and by returning to within that domain after having traversed the path $C$.

26

We will see that the general case can be brought back to the case examined. To this effect, suppose the projection of any one contour $C$ evaluated in relation to surface $S$ is determined by the equation

$$\sum x_i^2 = 1.$$

By putting

$$x_i' = \frac{x_i}{\sqrt{\sum x_i^2}}, \quad (i = 1, 2, \ldots, n) \tag{6}$$

one will call the point $(x_i')$ the projection of the point $(x_i)$ within the surface $S$.

Designate by $C'$ a projection of this contour $C$.

Suppose that by traversing the contour $C'$, one returns to the initial point $(\xi_i')$ with the same solution of the function $P(x_1, \ldots, x_n)$ of which when leaving that point. I argue that one will return to the corresponding point $\xi_i$ of the contour $C$ with the same solution of the function $P(x_1, \ldots, x_n)$.

To demonstrate this, it suffices to observe that the points $(\xi_i)$ and $(\xi_i')$, by virtue of equalities (6), belong to the same domains of the series (2).

One concludes that it suffices to examine the different closed contours belonging to the surface $S$.

27

Introduce in our study a function $d(x_i, x_i')$ being defined by the formula

$$d(x_i, x_i') = \sqrt{\sum (x_i' - x_i)^2}.$$

One will call distance between two points $(x_i)$ and $(x_i')$ the corresponding value of the function $d(x_i, x_i')$.

*Lemma. One can determine a positive parameter $\delta$ satisfying the following condition: every closed contour $C$ belonging to the surface $S$ will be situated in the domains which are contiguous by at least one edge, if the distance of all the point of the contour $C$, each of all to the rest, do not exceed the limit $\delta$.*

Let $(\xi_i)$ be a point of the contour $C$ belonging to the domain $A$. Put

$$\xi_i = \sum_{k=1}^{n} \rho_k a_{ik} \quad \text{where} \quad \rho_k \geq 0. \quad (k = 1, 2, \ldots, n)$$

By virtue of the equation

$$\sum \xi_i^2 = 1,$$

the sum $\sum_{k=1}^{n} \rho_k$ is not less than a positive fixed limit.

$$\sum_{k=1}^{n} \rho_k \geq \tau. \tag{7}$$

Suppose that the contour $C$ is not situated entirely within the domain $A$.

Let $(\xi_i')$ be a point of $C$ which does not belong to the domain A. By putting

$$\xi_i' = \sum_{k=1}^{n} \rho_k' a_{ik}, \tag{8}$$

one will have among the numbers $\rho_1', \rho_2', \ldots, \rho_n'$ at least one negtive number.

Suppose, to fix an ideas, that

$$\rho_1' \geq 0, \rho_2' \geq 0, \ldots, \rho_\mu' \geq 0 \tag{9}$$

and that

$$\rho_{\mu+1}' < 0, \rho_{\mu+2}' < 0, \ldots, \rho_n' < 0. \tag{10}$$

After the supposition made, one has the inequality

$$d(\xi_i, \xi_i') \geq \delta.$$

One can choose the parameter $\delta$, of such a manner that one had the inequalities

$$|\rho_k' - \rho_k| < \epsilon, \ (k = 1, 2, \ldots, n) \tag{11}$$

$\epsilon$ being a positive parameter also small as one would wish.

By (10), one obtains

$$0 \leq \rho_k < \epsilon, -\epsilon < \rho_k' < 0. \ (k = \mu+1; \mu+2, \ldots, n) \tag{12}$$

Choose among the numbers $\rho_1, \rho_2, \ldots, \rho_n$ the one which is the largest. By virtue of the inequality (7), this number can not be less than $\frac{\tau}{n}$. By supposing that

$$\epsilon < \frac{\tau}{n},$$

one will find the number looked for among the numbers $\rho_1, \rho_2, \ldots, \rho_\mu$. Suppose, to fix the ideas, that

$$\rho_1 > \frac{\tau}{n}.$$

the inequality (11) gives

$$\rho'_1 > \frac{\tau}{n} - \epsilon. \tag{13}$$

This posed, suppose that the point $(\xi'_i)$ belonged to the domain $A'$ and put

$$\xi'_i = \sum_{k=1}^{n} u_k a'_{ik} \text{ where } u_k \geq 0. \ (k = 1, 2, \ldots, n) \tag{14}$$

Designate by $(\alpha_i)$ and $\alpha'_i$ two vertices of the parallelohedron $R$ corresponding to domains $A$ and $A'$ by defining them by the equations

$$a_{0k} + \sum a_{ik} x_i = 0, \quad (k = 1, 2, \ldots, n)$$

and by the equations

$$a'_{0k} + \sum a'_{ik} x_i = 0. \ (k = 1, 2, \ldots, n) \tag{15}$$

By virtue of equality (8) and (14), one obtains an identity

$$\rho'_0 + \sum_{k=1}^{n} \rho'_k (a_{0k} + \sum a_{ik} x_i) = \sum_{k=1}^{n} u_k (a'_{0k} + \sum a'_{ik} x_i). \tag{16}$$

By making in this identity $x'_i = \alpha_i$, one finds

$$\rho'_0 = \sum u_k (a'_{0k} + \sum a'_{0k} \alpha_i) \geq 0. \tag{17}$$

By making in the identity (16) $x_i = \alpha'_i$, it will become

$$\rho'_0 + \sum_{k=1}^{n} \rho'_k (a_{0k} + \sum a_{ik} \alpha'_i) = 0. \tag{18}$$

Suppose that

$$a_{01} + \sum a_{i1} \alpha'_i > 0.$$

By virtue of (7), (12), (13) and (17), one will have

$$\rho'_0 + \sum_{k=1}^{\mu} \rho'_k (a_{0k} + \sum a_{ik} \alpha'_i) > (\frac{\tau}{n} - \epsilon)(a_{01} + \sum a_{i1} \alpha'_i),$$

$$\sum_{k=\mu+1}^{n} \rho'_k (a_{0k} + \sum a_{ik} \alpha'_i) \geq -\epsilon \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik} \alpha'_i),$$

and the equality (18) gives

$$\frac{\tau}{n}(a_{01} + \sum a_{i1}\alpha'_i) < \epsilon \left[ a_{01} + \sum a_{i1}\alpha'_i + \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha'_i) \right]. \tag{19}$$

Designate

$$A = \frac{\tau}{n}(a_{01} + \sum a_{i1}\alpha'_i) \text{ and } B = a_{01} + \sum a_{i1}\alpha'_i + \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha'_i),$$

one will have

$$A > 0 \text{ and } B > 0,$$

and as a result

$$\epsilon > \frac{A}{B}. \tag{20}$$

One could determine the ratio $\frac{A}{B}$ correspondent to the different vertices of the parallelohedron $R$. Designate by $\omega$ the smallest of these ratios which is not zero. The parameter $\epsilon$ being arbitrary, one can suppose that

$$\epsilon \leq \omega.$$

The inequality (20) becomes impossible, it is therefore necessary that $A = 0$ or [to put it] differently

$$a_{01} + \sum a_{i1}\alpha'_i = 0.$$

By virtue of the equality obtained, the coefficients of the equation

$$a_{01} + \sum a_{i1}x_i = 0$$

are proportional to those of an equation which is among the equations (15).

By putting

$$a_{01} + \sum a_{i1}x_i = u(a'_{0h} + \sum a'_{ih}x_i) \text{ where } u > 0,$$

one will have

$$a_{i1} = ua'_{ih}. \quad (i = 1, 2, \ldots, n)$$

We have arrived at the following result: all the domain traversed by the contour $C$ examined are contiguous by at least one edge which is characterised by the point $(a_{i1})$.
28

We are now in the state of reaching the demonstration of the fundamental theorem announced.

Let $C$ be any contour belonging to the surface $S$. Suppose that on leaving the point $(\xi_i)$ one passes via the points $(\xi_i^{(0)}), (\xi_i'), (\xi_i'')$ and one returns to the point $(\xi_i)$.

The path around the contour $C$ can be replaced by the paths $C^{(0)}$ and $C'$.

The contour $C^{(0)}$ will be composed of a group $(\xi_i) - (\xi_i^{(0)})$ of $C$, of the vector $[\xi_i^{(0)}, \xi_i'']$ and of a group $(\xi_i^{(0)}) - (\xi_i') - (\xi_i'')$ of the contour $C$ and of the vector $[\xi_i'', \xi_i^{(0)}]$.

Suppose that by traversing the paths $C^{(0)}$ and $C'$ one uniformly defined the function $P(x_1, x_2, \ldots, x_n)$. In this case the trajectory by the group $(\xi_i^{(0)}) - (\xi_i')$ of the contour $C$ can be replace by the path the length of the vector $[\xi_i^{(0)}, \xi_i'']$.

By replacing the group $(\xi_i^{(0)}) - (\xi_i') - (\xi_i'')$ of the contour $C$ by the vector $[\xi_i^{(0)}, \xi_i'']$, one will transform the contour $C$ to $C^{(0)}$, thus, by traversing the contour $C$, one will return to the point $(\xi_i)$, by virtue of suppositions made, with the same solution of the function $P(x_1, x_2, \ldots, x_n)$.

Two contours $C^{(0)}$ and $C'$ can be examined in the same manner and so on.

Suppose that one had determined the contours

$$C_1, C_2, \ldots, C_m \tag{21}$$

which replace the path $C$. By supposing that the function $P(x_1, x_2, \ldots, x_n)$ be uniform the length of contour (21), one will demonstrate that it will be uniform the length of the contour $C$ given.

This established, observe that we can always choose the contours (21), of such a manner that their contours satisfy the conditions of the lemma of the previous Number. In this case, any contour (21) will be situated within domains which are contiguous by at least one edge. We have seen in Number 25 that by traversing the same contours one will always return to the point of departure by the same solution of the function $P(x_1, x_2, \ldots, x_n)$ as while leaving this point. It is thus demonstrated that any closed contour $C$ possesses the same property.

We have demonstrated that the function $P(x_1, x_2, \ldots, x_n)$ is uniformly defined in any domain of the set (1) (Number 22). It remains to demonstrate that the function $P(x_1, x_2, \ldots, x_n)$ is well defined in any point of the space in $n$ dimensions.

Suppose that a point $\xi_i$ belongs to two domains $A$ and $A^{(0)}$.

I argue that the function $P(x_1, x_2, \dots, x_n)$ for the point $\xi_i$ will have one same value in the domain $A$ and in the domain $A^{(0)}$.

To demonstrate this, one will form a series of domains

$$A, A', \dots, A^{(m)}, A^{(0)}$$

which are successively contiguous by faces in $n-1$ dimensions and in which belongs the point $\xi_i$.

As the point $\xi_i$ belongs to the face common to domains $A$ and $A'$, one will have ,by virtue of the formula (5) of Number 23,

$$P_{(A)}(\xi_1, \xi_2, \dots, \xi_n) = P_{(A')}(\xi_1, \xi_2, \dots, \xi_n).$$

In the same manner, one obtains

$$P_{(A')}(\xi_1, \xi_2, \dots, \xi_n) = P_{(A'')}(\xi_1, \xi_2, \dots, \xi_n),$$
$$\dots$$
$$P_{(A^{(m)})}(\xi_1, \xi_2, \dots, \xi_n) = P_{(A^{(0)})}(\xi_1, \xi_2, \dots, \xi_n).$$

It results in that

$$P_{(A)}(\xi_1, \xi_2, \dots, \xi_n) = P_{(A^{(0)})}(\xi_1, \xi_2, \dots, \xi_n).$$

The fundamental theorem announced is thus demonstrated.

*Canonical form of inequalities which define the set* $(R)$ *of primitive parallelohedron* $S$.

29

Choose within the set $(R)$ of primitive parallelohedra any parallelohedron $R_0$ . Suppose that the parallelohedron $R_0$ is determined with the help of canonical inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

Observe that we can replace these inequalities by the following canonical inequalities:

$$u(a_{0k} + \sum a_{ik} x_i) \geq 0, \quad (k = 1, 2, \dots, \sigma)$$

$u$ being a positive arbitrary parameter.

Designate by $R_k$ $(k = 1, 2, \dots, \sigma)$ the parallelohedron which is contiguous to the parallelohedron $R_0$ by the face determined within $R_0$ by the equation

$$a_{0k} + \sum a_{ik} x_i = 0 \qquad (1)$$

and suppose that the vector $[\lambda_{ik}]$ defined a translation of the parallelohedron $R_k$ to $R_0$.

It follows that the parallelohedron $R_k$will be determined by the canonical inequalities

$$a_{0h} + \sum a_{ih}(x_i + \lambda_{ik}) \geq 0, \quad (h = 1, 2, \ldots, \sigma)$$

or by the canonical inequalities

$$u_k[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})] \geq 0, \ (h = 1, 2, \ldots, \sigma) \tag{2}$$

$u_k$ being an arbitrary positive parameter.

The face $P_k$ in $n-1$ dimensions common to the parallelohedra $R$ and $R_k$ is defined in the parallelohedron $R_0$ by the equation (1). Within the parallelohedron $R_k$, the face $P_k$ will be determined by an equation in which the coefficients are proportional to those of the equation

$$-a_{0k} - \sum a_{ik}x_i = 0.$$

One can choose the positive parameter $u_k$, of a manner such that one had the identity

$$-a_{0k} - \sum a_{ik}x_i = u_k(a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})).$$

In this case, the inequality

$$-a_{0k} - \sum a_{ik}x_i \geq 0$$

is found among the inequalities (2) which define the parallelohedron $R_k$.

One will say that these inequalities are represented in the canonical form.

30

Observe an important property of canonical inequalities which define the parallelohedra $R_0, R_1, R_2, \ldots, R_\sigma$.

Let $(\alpha_i)$ be a vertice of the parallelohedron $R_0$ determined by the canonical equations

$$a_{0k} + \sum a_{ik}x_i = 0. \ (k = 1, 2, \ldots, n) \tag{3}$$

Examine the canonical equations which define the vertice $(\alpha_i)$ in the parallelohedron $R_k$ $(k = 1, 2, \ldots, n)$.

The equations (3) being canonical, one will determine the vertice $(\alpha_i)$ within the parallelohedron $R_k$, by virtue of the theorem of

Number 19, by the equation

$$\sum (a_{ih} - a_{ik})(x_i - \alpha_i) = 0, \quad (h = 1, 2, \ldots, n, h \neq k)$$

$$-\sum a_{ik}(x_i - \alpha_i) = 0 \quad (k = 1, 2, \ldots, n)$$

By virtue of the supposition made, the inequality

$$-\sum a_{ik}(x_i - \alpha_i) \geq 0$$

exists among the canonical inequalities (2) which define the parallelohedron $R_k$, which results in that the inequalities

$$\sum (a_{ih} - a_{ik})(x_i - \alpha_i) \geq 0, \quad (h = 1, 2, \ldots, n, h \neq k)$$

also exist among the canonical inequalities (2).

One concludes that the canonical equation

$$\sum (a_{ih} - a_{ik})(x_i - \alpha_i) = 0$$

define in the parallelohedron $R_k$ a face in $n-1$ dimensions which is common to the parallelohedra $R_k$ and $R_h$. the same face will be determined in the parallelohedron $R_h$ by a canonical equation

$$\sum (a_{ik} - a_{ih})(x_i - \alpha_i) = 0.$$

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by applying the procedure explained, one can determine the canonical inequalities which define the parallelohedra contiguous to the parallelohedra $R_1, R_2, \ldots, R_\sigma$ and so on.

For every parallelohedron $R$ of the set $(R)$, one can form a series of parallelohedra

$$R_0, R', R'', \ldots, R^{(m)}, R$$

which are successively contiguous. One will determine successively the canonical inequalities that define the parallelohedra of this series.

One could arrive at the parallelohedron $R$ by other ways and determine the canonical inequalities which define the parallelohedron $R$ in various manners.

We shall se that the canonical inequalities which define a parallelohedron of the set $(R)$ do not depend on the path by which one arrives at the parallelohedron $(R)$ leaving from the principal parallelohedron $R_0$.

### *Generatrix function of the set $(R)$ of primitive parallelohedra.*

32

Consider a set $(R)$ of primitive parallelohedra. Suppose any parallelohedron $R$ of the set $(R)$ be characterised by a vector $[\lambda_i]$ which defines a translation of parallelohedra $R$ to a principal parallelohedron $R_0$.

Designate by $G$ the group of vectors $[\lambda_i]$ which correspond to the different parallelohedra of the set $(R)$.

Introduce in our study a function

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$$

of variables $x_1, x_2, \ldots, x_n$ and parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ by defining it within the space in $n$ dimensions and for the group $G$ such that:

1. Within the principal parallelohedron $R_0$, one will write

$$V(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0,) = 0.$$

2. Within the parallelohedron $R_k$ which is contiguous to $R_0$, one will write

$$V(x_1, x_2, \ldots, x_n, \lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{nk} = a_{0k} + \sum a_{ik} x_i, \ (k = 1, 2, \ldots, \sigma)$$

providing that in the parallelohedron $R_0$ the canonical equation

$$a_{0k} + \sum a_{ik} x_i = 0$$

had the face in $n-1$ dimensions common to the parallelohedra $R_0$ and $R_k$.

3. By supposing that the parallelohedra $R$ and $R'$ characterised by the vectors $[\lambda_i]$ and $\lambda'_i$ are contiguous by a face in $n-1$ dimensions which is defined within $R$ by a canonical equation

$$a_0 + \sum a_i x_i = 0,$$

one will write

$$V(x_1, x_2, \ldots, x_n, \lambda_1,', \lambda'_2, \ldots, \lambda'_n) =$$
$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) + a_0 + \sum a_i x_i.$$

33

Let $R$ be any parallelohedron of the set $(R)$ characterised by a vector $[\lambda_i]$. One will form a series of parallelohedra.

$$R_0, R', \ldots, R^{(m)}, R$$

which are successively contiguous by faces in $n-1$ dimensions. Designate by

$$a_0^{(0)} + \sum a_i^{(0)} x_i = 0$$

the equation of the faces common to the parallelohedra $R_0$ and $R'$ and defined in $R_0$; designate by

$$a'_0 + \sum a'_i x_i = 0$$

the equaion of the face common to the parallelohedra $R'$ and $R''$ defined in $R'$ and so on.

By applying the definition established, one will determine the function

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$$

by the formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i).$$

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*Fundamental theorem. The function* $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ *is well defined for any vector* $[\lambda_i]$ *of the group* $G$.

Suppose that one had formed a series of parallelohedra

$$R, R', R'', \ldots, R^{(m)}, R \qquad (1)$$

which are successively contiguous. On leaving the parallelohedron $R$ with any solution of the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$, one will return inside the parallelohedron $R$ after having traversed the parallelohedra (1) with a solution of the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ which, by virtue of the definition established, is expressed by the vertice

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) + \sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i).$$

We shall demonstrate that one will always have

$$\sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i) = 0.$$

Examine, in the first place, the case where all the parallelohedra (1) are contiguous by at least one vertice $(\alpha_i)$. By virtue of Theorem II of Number 17, all the primitive parallelohedra (1) will be in this case contiguous one to one through faces in $n-1$ dimensions.

Designate by

$$a_{0k} + \sum a_{ik}(x_i - \alpha_i) = 0, \quad (k = 1, 2, \ldots, m)$$

the canonical equation of the face common to the parallelohedra $R^{(k)}$ and $R\,(k=1,2,\dots,m)$ defined within the parallelohedron $R$.

We have seen in Number 30 that the canonical equation of the face common to the parallelohedra $R'$ and $R''$ and defined within $R'$ will be

$$\sum(a_{i2}-a_{i1})(x_i-\alpha_i)=0$$

and so on and so forth. One obtains the formulae

$$\begin{aligned}
a_0^{(0)}+\sum a_i^{(0)}x_i&=\sum a_{i1}(x_i-\alpha_i),\\
a_0'+\sum a_i'x_i&=\sum(a_{i2}-a_{i1})(x_i-\alpha_i),\\
&\dots\\
a_0^{(m-1)}+\sum a_i^{(m-1)}x_i&=\sum(a_{im}-a_{i,m-1})(x_i-\alpha_i),\\
a_0^{(m)}+\sum a_i^{(m)}x_i&=-\sum a_{im}(x_i-\alpha_i),
\end{aligned}$$

and it follows that

$$\sum_{k=0}^{m}(a_0^{(k)}+\sum a_i^{(k)}x_i)=0.$$

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We shall see that the general case can be brought back to the case examined.

*Theorem. One can determine a positive parameter $\delta$, in a manner that every closed contour $C$ is found within the parallelohedra which are contiguous by at least one vertex, providing that the distance between any two points of the contour $C$ does not exceed the limit $\delta$.*

Observe, in the first place, that the distance between the points $(\xi_i)$ and $(\xi_i')$ belonging to the two parallelohedra which are not contiguous can not be less than a fixed limit. To demonstrate this, suppose that the point $(\xi_i)$ belong to the parallelohedron $R$ defined with the help of inequalities

$$a_{0k}+\sum a_{ik}x_i\geq 0.\quad (k=1,2,\dots,\sigma)$$

Designate by $R_1, R_2, \dots, R_\sigma$ the parallelohedra which are contiguous to $R$ and examine the set $K$ of points belonging to the parallelohedra $R_1, R_2, \dots, R_\sigma$.

Designate by

$$(\alpha_{i1}),(\alpha_{i2}),\dots,(\alpha_{is})$$

the vertices of parallelohedron $R$ and designate by

$$(\alpha_{i1}^{(h)}), (\alpha_{i2}^{(h)}), \ldots, (\alpha_{is}^{(h)}), \quad (h = 1, 2, \ldots, \sigma)$$

the vertices of parallelohedron $R_h$ $(h = 1, 2, \ldots, \sigma)$.

By virtue of the supposition made, one will have the inequalities

$$a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \leq 0, \ (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma).$$

Designate by $\rho$ the smallest numerical value of vertices

$$a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \quad (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma)$$

which does not become zero. By virtue of supposition made, one will have the inequality

$$\rho + a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \leq 0,$$

on condition that

$$a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} < 0,$$

where $(k = 1, 2, \ldots, s,\ h = 1, 2, \ldots, \sigma)$.

This established, take any point $(\xi_i')$ which does not belong to the set $K$. Examine the points of a vector $[\xi_i, \xi_i']$. By putting

$$x_i = \xi_i + u(\xi_i' - \xi_i) \text{ where } 0 \leq u \leq 1,$$

let us think the parameter $u$ of a continuous manner within the interval $0 < u < 1$. One will determine a point

$$\xi_i^{(0)} = \xi_i + u_0(\xi_i' - \xi_i) \text{ where } 0 < u_0 < 1 \tag{2}$$

which belongs to the boundary of the set $K$, that is to say to a face in $n-1$ dimensions of parallelohedra $R_1, R_2, \ldots, R_\sigma$ and which also belongs to another parallelohedron $R'$.

Suppose that the point $(\xi_i^{(0)})$ belongs to the parallelohedron $R_h$. The parallelohedra $R_h$ and $R'$ will be contiguous by a face in $n-1$ dimensions.

Designate by

$$(\alpha_{i1}^{(h)}), (\alpha_{i2}^{(h)}), \ldots, (\alpha_{it}^{(h)}) \tag{3}$$

the vertices of parallelohedron $R_h$ which belong to this face.

None of these vertices verifies the equation

$$a_{0h} + \sum a_{ih}x_i = 0$$

because otherwise the face examined would belong to two parallelohedron of the series $R, R_1, \ldots, R_\sigma$, which is contrary to the hypothesis.

Therefore one will have the inequalities

$$\rho + a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \leq 0. \quad (k = 1, 2, \ldots, t)$$

The point $(\xi_i^{(0)})$ belonging to the face of $R_h$, which is characterised by the vertices (3), probably determined by the equations

$$\xi_i^{(0)} = \sum_{k=1}^{k=t} \vartheta_k \alpha_{ik}^{(h)} \text{ where } \sum_{k=1}^{k=t} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0. \, (k = 1, 2, \ldots, t)$$

Of the previous inequalities, one draws

$$\rho + a_{0h} + \sum a_{ih}\xi_i^{(0)} \leq 0.$$

By observing that on the other hand one has

$$a_{0h} + \sum a_{ih}\xi_i \geq 0, \tag{4}$$

one finds, by (2),

$$\rho + a_{0h} + \sum a_{ih}\xi_i' < 0. \tag{5}$$

By virtue of inequalities (4) and (5), the distance $d(\xi_i, \xi_i')$ can not be smaller than a fixed limit $d$.

### *Solution of the centre of the primitive parallelohedra*

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This established, examine a contour $C$ formed which the points had the mutual distance that does not surpass $\delta$ . By supposing that

$$\delta < d,$$

one will have a contour $C$ which is situated within the contiguous parallelohedra two to two. [one to one]

Let $\xi_i$ be any point of the contour $C$ belonging to the parallelohedron $R$. Suppose that not all the points of contour $C$ belong to $R$ and designate by $\xi_i'$ a point of contour $C$ which does not belong to $R$. Put

$$\xi_i = \sum_{k=1}^{s} \vartheta_k \alpha_{ik} \text{ where } \sum \vartheta_k = 1 \text{ and } \vartheta_k \geq 0, \, (k = 1, 2, \ldots, s) \tag{6}$$

$$\xi_i' = \sum_{k=1}^{s} \vartheta_k' \alpha_{ik} \text{ where } \sum \vartheta_k' = 1 \tag{7}$$

As the point $(\xi_i')$ does not belong to $R$, one will have among the numbers $\vartheta_1', \vartheta_2', \ldots, \vartheta_s'$ at least one number which will be negative. Suppose, to fix the ideas, that

$$\vartheta_1' \geq 0, \ldots, \vartheta_\mu' \geq 0 \text{ and } \vartheta_{\mu+1}' < 0, \ldots, \vartheta_s' < 0 \tag{8}$$

One can choose the parameter $\delta$ as small that one would have the inequalities

$$|\vartheta_k' - \vartheta_k| < \epsilon, \quad (k = 1, 2, \ldots, s) \tag{9}$$

$\epsilon$ being a positive parameter also as small as one would like. By virtue of (8), it will become

$$0 \leq \vartheta_k < \epsilon, \ -\epsilon < \vartheta_k' < 0. \quad (k = \mu + 1, \ldots, s) \tag{10}$$

Observe how the largest one among the numbers $\vartheta_1, \vartheta_2, \ldots, \vartheta_s$ can not be smaller than $\frac{1}{s}$ by (6) by supposing that

$$\epsilon < \frac{1}{s},$$

one wil find the required number among the number $\vartheta_1, \vartheta_2, \ldots, \vartheta_\mu$ . Suppose, for fixing ideas, that

$$\vartheta_1 > \frac{1}{s}.$$

By virtue of (9), it will become

$$\vartheta_1' > \frac{1}{s} - \epsilon \tag{11}$$

We have demonstrated that the point $(\xi_i')$ can not belong to these parallelohedra $R_1, R_2, \ldots, R_\sigma$ which are contiguous to $R$. By supposing that the point $(\xi_i')$ belong to the parallelohedron $R_h$, one will have an equality

$$a_{0h} + \sum a_{ih}\xi_i' < 0 \tag{12}$$

Observing that by virtue of equations (7)

$$a_{0h} + \sum a_{ih}\xi_i' = \sum_{k=1}^{s} \vartheta_k'(a_{0h} + \sum a_{ih}\alpha_{ik}),$$

one obtains, because of (12),

$$\sum_{k=1}^{s} \vartheta_k'(a_{0h} + \sum a_{ih}\alpha_{ik}) < 0$$

Of this inequality one draws, by (10) and (11),

$$\frac{1}{2}(a_{0h}\sum a_{ih}\alpha_{i1}) - \epsilon\left[a_{0h}\sum a_{ih}\alpha_{i1} + \sum_{k=\mu+1}^{s}(a_{0h} + \sum a_{ih}\alpha_{ik})\right] < 0\,.$$

By putting

$$A = \frac{1}{2}(a_{0h}\sum a_{ih}\alpha_{i1}) \text{ and } B = a_{0h}+\sum a_{ih}\alpha_{i1} + \sum_{k=\mu+1}^{s}(a_{0h}+\sum a_{ih}\alpha_{ik}),$$

suppose that $A > 0$; the previous inequality gives $B > 0$, thus

$$\epsilon > \frac{A}{B} \tag{13}$$

Observe that the numbers $A$ and $B$ do not change when one replace the parallelohedron by any parallelohedron of the set $(R)$. One concludes that the ratio $\frac{A}{B}$ which does not vanish possess a positive minimum $\omega$.

By supposing that

$$\epsilon < \omega,$$

the inequality (13) becomes impossible and it is necessary that $A = 0$ or otherwise

$$a_{0h} + \sum a_{ih}\alpha_{i1} = 0$$

We have arrived at the following result: all the parallelohedra within which is situated the contour examined $C$ are contiguous by the vertice $(\alpha_{i1})$.

With the help of the lemma of Number 35, one will easily demonstrate the fundamental theorem stated by repeating the reasoning explained in Number 28.

*Fundamental properties of the generatrix function* $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$

37

*Theorem I. Suppose that two vectors* $[\lambda_i]$ *and* $[\lambda_i^{(0)}]$ *characterise two parallelohedra* $R$ *and* $R^{(0)}$ *of the set* $(R)$. *One will have an inequality*

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) > V\left(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}\right),$$

*on condition that the point* $(x_i)$ *be interior to the parallelohedron* $R^{(0)}$.

Let$(\xi_i^{(0)})$ be any point which is interior to the parallelohedron $R^{(0)}$.

Take a point $(\xi_i)$ which is interior to the parallelohedron $R$ and examine a vector $[\xi_i^{(0)}, \xi_i]$ determined by the equations

$$x_i = \xi_i^{(0)} + u(\xi_i - \xi_i^{(0)}) \quad \text{where } 0 \leq u \leq 1\,.$$

A group of the vector $[\xi_i^{(0)}, \xi_i]$ belong to the parallelohedron $R^0$. Designate,

$$\xi_i' = \xi_i^{(0)} + u_1(\xi_i - \xi_i^{(0)}) \quad \text{where } 0 < u < 1$$

and suppose that the vector $[\xi_i^{(0)}, \xi_i']$ represents the group of the vector $[\xi_i^{(0)}, \xi_i]$ which belong to $R^{(0)}$.

The second group $[\xi_i', \xi_i]$ of the vector $[\xi_i^{(0)}, \xi_i]$ does not possess any point $(\xi_i')$ common to the parallelohedron $R^{(0)}$. The point $(\xi_i')$ belongs to a face $p^{(0)}(\nu)$ of the parallelohedron $R^{(0)}$. One will choose among the parallelohedra which are contiguous by the face $p^{(0)}(\nu)$ a parallelohedron $R'$ which contains a group of the vector $[\xi_i', \xi_i]$.

Designate

$$\xi_i'' = \xi_i + u_2(\xi_i - \xi_i^{(0)}) \quad \text{where } u_1 < u_2 \leq 1$$

and suppose that the vector $[\xi_i', \xi_i'']$ represents a group of the vector $[\xi_i', \xi_i]$ which belongs to the parallelohedron $R'$ and so on.

Let us suppose that one has determined $m$ points of the vector $[\xi_i^{(0)}, \xi_i]$

$$\xi_i^{(k)} = \xi_i^{(0)} + u_k(\xi_i - \xi_i^{(0)})\,, \; (k = 1, 2, \ldots, m) \tag{1}$$

where

$$0 < u_1 < u_2 < \cdots < u_m < 1 \tag{2}$$

which correspond to the vectors $[\xi_i^{(0)}, \xi_i'], [\xi_i', \xi_i''], \ldots, [\xi_i^{(m)}, \xi_i]$ belonging to the parallelohedra

$$R^{(0)}, R', \ldots, R^{(m-1)}, R$$

successively contiguous.

Designate by

$$a_0^{(k)} + \sum a_i^{(k)} x_i = 0, \quad (k = 0, 1, 2, \ldots, m-1)$$

the canonical equation of the face common to the parallelohedra $R^{(k)}$ and $R^{(k+1)}$ which is defined within the parallelohedron $R^{(k)}$.

By virtue of the established definition in Number 32, one will have a formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = V\left(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}\right) + \sum_{k=0}^{m-1}\left(a_0^{(k)} + \sum a_i^{(k)} x_i\right) \tag{3}$$

Examine the sum

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(0)} \text{ and } a_0^{(k)} + \sum a_i^{(k)} \xi_i. \quad (k = 0, 1, 2, \ldots, m-1)$$

By virtue of the supposition made, the point $(\xi_i^{(k+1)})$ verifies the equation

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(k+1)} = 0$$

As the point $(\xi^{(k)})$ belongs to the parallelohedron $R^{(k)}$, one will have an inequality

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(k)} \geq 0 \,.$$

By virtue of (1) and (2), one obtains

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(0)} \geq 0 \text{ and } a_0^{(k)} + \sum a_i^{(k)} \xi_i \leq 0 \,. \; (k = 0, 1, 2, \ldots, m-1)$$

As the point $(\xi_i^{(0)})$ is interior to the parallelohedron $R^{(0)}$, we will have

$$a_0^{(0)} + \sum a_i^{(0)} \xi_i^{(0)} > 0 \text{ and } a_0^{(0)} + \sum a_i^{(0)} \xi_i < 0 \,,$$

It results in that

$$\sum_{k=0}^{m-1} (a_0^{(0)} + \sum a_i^{(0)} \xi_i^{(0)}) > 0 \text{ and } \sum_{k=0}^{m-1} (a_0^{(k)} + \sum a_i^{(k)} \xi_i) < 0 \,.$$

By substituting in the formula (3), one obtains

$$V(\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_n^{(0)}, \lambda_1, \lambda_2, \ldots, \lambda_n) > V(\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_n^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)})$$

and

$$V(\xi_1, \xi_2, \ldots, \xi_n, \lambda_1, \lambda_2, \ldots, \lambda_n) < V(\xi_1, \xi_2, \ldots, \xi_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}) \,.$$

*Theorem II. Suppose that the parallelohedra* $R^{(0)}, R', \ldots, R^{(n-\nu)}$ *be contiguous by a face* $p(\nu)$ *in* $\nu$ *dimensions. By designating by* $[\lambda_i^{(k)}]$, $(k = 0, 1, 2, \ldots, n-\nu)$ *the vectors which characterise these parallelohedra, one will have an inequality*

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) > V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}),$$

*on condition that the point* $(x_i)$ *be interior to a face* $p(\nu)$ *and that the vector* $[\lambda_i]$ *is not among the vectors* $[\lambda_i^{(k)}]$, $(k = 1, 2, \ldots, (n-\nu))$.

By supposing that $\lambda_i = \lambda_i^{(k)}$, one will have the equation

$$V(x_1, x_2, \ldots, x_n, \lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_n^{(k)}) = V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}).$$
$$(k = 1, 2, \ldots, (n-\nu))$$

One will easily demonstrate the announced Theorem II by repeating the reasonings which have been established previously.

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The results obtained open a new way for the researches concerning the primitive parallelohedra. One can consider the set $(R)$ of primitive parallelohedra under a new point of view, in knowing:

*Each parallelohedron* $R^{(0)}$ *of the set* $(R)$ *characterised by the vector* $[\lambda_i^{(0)}]$ *presents a set of points* $(x_i)$ *verifying the inequality* $V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) \geq V(x_i, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)})$, *for any vector* $[\lambda_i]$ *belonging to group* $G$.

We have seen in Number 32 that for the principal parallelohedron $R_0$ of the set $(R)$ one has

$$V(x_i, x_2, \ldots, x_n, 0, 0, \ldots, 0) = 0.$$

It follows that the principal parallelohedron $R_0$ is defined by the inequality

$$V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) \geq 0$$

which holds for any vector $[\lambda_i]$ of group $G$.

### *Solution of the quadratic function* $V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$

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Suppose that the principal parallelohedron $R_0$ is determined with the help of canonical inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Designate by $[\lambda_{ik}]$ the vector which defines a translation of the parallelohedron $R_k$ to $R_0$ $(k = 1, 2, \ldots, \sigma)$.

Take two parallelohedra $R_k$ and $R_0$ contiguous to the parallelohedron $R_0$ through the faces $P_k$ and $P_h$ which are not parallel. Put

$$\lambda = \lambda_k + \lambda_{ih}$$

and designate by $R$ the parallelohedron of the set $(R)$ characterised by the vector $[\lambda_i]$.

The parallelohedron $R$ is contiguous to the parallelohedra $R_k$ and $R_h$ through the faces which are congruent to the faces $P_h$ and $P_k$.

One can thus form the series

$$R_0, R_k, R \text{ and } R_0, R_h, R$$

of parallelohedra which are successively contiguous.

Let us suppose that the parallelohedron $R_k$ is determined with the help of canonical equations

$$u_k[a_{0r} + \sum a_{ir}(x_i + \lambda_{ik})] \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

The face of the parallelohedron $R_k$ which is congruent to the face $P_h$ will be determined by the equation

$$u_k[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})] = 0.$$

It results in that the function $V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is expressed by the sum

$$V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = a_{0k} + \sum a_{ik}x_i + u_k\left[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})\right].$$

In the same manner, one obtains

$$V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = a_{0h} + \sum a_{ih}x_i + u_h\left[a_{0k} + \sum a_{ik}(x_i + \lambda_{ih})\right].$$

By virtue of the fundamental theorem of Number 34, one will have an identity

$$a_{0k} + \sum a_{ik}x_i + u_k\left[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})\right] =$$
$$a_{0h} + \sum a_{ih}x_i + u_h\left[a_{0k} + \sum a_{ik}(x_i + \lambda_{ih})\right].$$

It follows that

$$a_{0k} + u_k(a_{0h} + \sum a_{ih}\lambda_{ik}) = a_{0h} + u_h(a_{0k} + \sum a_{ik}\lambda_{ih}) \qquad (1)$$

and

$$a_{ik} + u_k a_{ih} = a_{ih} + u_h a_{ik}. \quad (i = 1, 2, \ldots, n)$$

We have supposed that the coefficients $a_{ik}$ and $a_{ih}$, $(i = 1, 2, \ldots, n)$ would not be proportional , thus it is necessary that

$$u_k = 1 \text{ and } u_h = 1$$

We have arrived at the following important result:

it Any parallelohedron $R$ characterised by a vector $[\lambda_i]$ will be determined by the canonical inequalities

$$a_{0k} + \sum a_{ik}(x_i + \lambda_i) \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

Observe that by virtue of (1), one will have the equation

$$\sum a_{ik}\lambda_{ih} = \sum a_{ih}\lambda_{ik}.$$

In this equation, one can attribute to the indices $k$ and $h$ the values $k = 1, 2, \dots, \sigma; h = 1, 2, \dots, \sigma$.

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*Theorem. The vectors*

$$[\lambda_{i1}], [\lambda_{i2}], \dots, [\lambda_{i\sigma}]$$

*form the basis of the group $G$. By posing*

$$\lambda_i = \sum_{k=1}^{\sigma} l_k \lambda_{ik} \tag{2}$$

*where $l_1, l_2, \dots, l_\sigma$ are arbitrary integers, one will determine each vector $[\lambda_i]$ of the group $G$. By indicating*

$$a_i = \sum_{k=1}^{\sigma} l_k a_{ik}, \tag{3}$$

*one will define the function $V(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_n)$ by the formula*

$$V(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{k=1}^{\sigma} l_k (a_{0k} - \frac{1}{2}\sum_{i=1}^{n} a_{ik}\lambda_{ik} + \sum_{i=1}^{n} a_{ik}x_i) + \frac{1}{2}\sum_{i=1}^{n} a_i\lambda_i \tag{4}$$

Let us suppose that the formula (4) is verified by the vectors $[\lambda_i^{(0)}]$ and $[\lambda_i']$ which are defined by the equations

$$\lambda_i^{(0)} = \sum_{k=1}^{n} l_k^{(0)} \lambda_{ik} \text{ and } \lambda_i' = \sum_{k=1}^{n} l_k' \lambda_{ik}. \ (i = 1, 2, \dots, n) \tag{5}$$

We will see that the formula (1) will also be true for the vector $[\lambda_i]$ determined by the equations

$$\lambda_i = \lambda_i^{(0)} + \lambda_i'.$$

Let us indicate by $R, R^{(0)}$ and $R'$ the parallelohedra characterised by the vectors $[\lambda_i], [\lambda_i^{(0)}]$ and $[\lambda_i']$.

The parallelohedron $R^{(0)}$ will be determined by the canonical inequalities

$$a_{0k} + \sum a_{ik}(x_i + \lambda_i^{(0)}) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One concludes that the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is expressed by the formula

$$\begin{gathered} V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \\ V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}) + U(x_1, x_2, \ldots, x_n, \lambda', \lambda_2', \ldots, \lambda_n'), \end{gathered} \tag{6}$$

Where the function $U(x_1, x_2, \ldots, x_n, \lambda', \lambda_2', \ldots, \lambda_n')$ represents the generatrix function determined with the condition that the parallelohedron $R^{(0)}$ have been chosen for the principal parallelohedron.

By designating

$$a_i^{(0)} = \sum_{k=1}^{\sigma} l_k^{(0)} a_{ik} \text{ and } a_i' = \sum_{k=1}^{\sigma} l_k' a_{ik}, \ (i = 1, 2, \ldots, n) \tag{7}$$

one will have, by virtue of the supposition made,

$$\begin{gathered} V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}) = \\ \textstyle\sum_{k=1}^{\sigma} l_k^{(0)} \left(a_{0k} - \frac{1}{2} \sum a_{ik}\lambda_{ik} + \sum a_{ik} x_i\right) + \frac{1}{2} \sum a_i^{(0)} \lambda_i^{(0)}, \end{gathered}$$

$$\begin{gathered} U(x_1, x_2, \ldots, x_n, \lambda', \lambda_2', \ldots, \lambda_n') = \\ \textstyle\sum_{k=1}^{\sigma} l_k' \left(a_{0k} + \sum a_{ik}\lambda_i^{(0)} - \frac{1}{2} \sum a_{ik}\lambda_{ik} - \sum a_{ik} x_i\right) + \frac{1}{2} \sum a_i' \lambda_i'. \end{gathered}$$

Let us put

$$l_k = l_k^{(0)} + l_k'. \quad (k = 1, 2, \ldots, \sigma)$$

By virtue of (6) one obtains

$$\begin{aligned} V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) &= \textstyle\sum l_k \left(a_{0k} - \frac{1}{2} \sum a_{ik}\lambda_{ik} + \sum a_{ik} x_i\right) \\ &\quad \textstyle+ \frac{1}{2} \sum a_i^{(0)} \lambda_i^{(0)} + \frac{1}{2} \sum a_i' \lambda_i' + \sum_{k=1}^{\sigma} \sum_{i=1}^{\sigma} a_{ik} l_k' \lambda_i^{(0)}. \end{aligned} \tag{8}$$

Let us examine the sum

$$\frac{1}{2} \sum a_i^{(0)} \lambda_i^{(0)} + \frac{1}{2} \sum a_i' \lambda_i' + \sum_{k=1}^{\sigma} \sum_{i=1}^{n} a_{ik} l_k' \lambda_i^{(0)}. \tag{9}$$

By virtue of (5), one will hve

$$\sum_{i=1}^{n} a_{ik} \lambda_i^{(0)} = \sum_{h=1}^{\sigma} \sum_{i=1}^{n} a_{ik} l_h^{(0)} \lambda_{ih}.$$

We have seen in Number 39 that

$$\sum_{i=1}^{n} a_{ik}\lambda_{ih} = \sum_{i=1}^{n} a_{ih}\lambda_{ik},$$

therefore

$$\sum_{i=1}^{n} a_{ik}\lambda_i^{(0)} = \sum_{h=1}^{\sigma}\sum_{i=1}^{n} a_{ih} l_h^{(0)} \lambda_{ik}$$

and, because of (7), this becomes

$$\sum_{i=1}^{n} a_{ik}\lambda_i^{(0)} = \sum_{i=1}^{n} a_i^{(0)} \lambda_{ik}.$$

It follows that

$$\sum_{k=1}^{\sigma}\sum_{i=1}^{n} a_{ik} l'_k \lambda_i^{(0)} = \sum a_i^{(0)} \lambda'_i.$$

By virtue of (7), one will also have

$$\sum_{k=1}^{\sigma}\sum_{i=1}^{n} a_{ik} l'_k \lambda_i^{(0)} = \sum_{i=1}^{n} a'_i \lambda_i^{(0)}.$$

One can therefore present the sum (9) under the form

$$\begin{aligned}\frac{1}{2}\sum a_i^{(0)}\lambda_i^{(0)} + \frac{1}{2}\sum a'_i\lambda'_i + \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik} l'_k \lambda_i^{(0)}& \\ = \frac{1}{2}\left(\sum a_i^{(0)}\lambda_i^{(0)} + \sum a_i^{(0)}\lambda'_i + \sum a'_i\lambda_i^{(0)} + \sum a'_i\lambda'_i\right)& \\ = \frac{1}{2}\sum \Big(a_i^{(0)} + a_i^{(0)})(\lambda_i^{(0)} + \lambda'_i\Big)\,.&\end{aligned}$$

As

$$a_i^{(0)} + a'_i = a_i \text{ and } \lambda_i^{(0)} + \lambda'_i = \lambda_i,$$

the formula (8) can be written

$$\begin{aligned}&V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \\ &\qquad \sum_{k=1}^{\sigma} l_k \left(a_{0k} - \frac{1}{2}\sum a_{ik}\lambda_{ik} + \sum a_{ik}x_i\right) + \frac{1}{2}\sum a_i\lambda_i.\end{aligned}$$

It is easy to verify the formula (4) in the case

$$\lambda_i = \pm\lambda_{ik}. \quad (k = 1, 2, \ldots, \sigma)$$

This results in that the formula (4) holds for any vector $[\lambda_i]$ belonging to the group $G$.

*Theorem II. The group $G$ possesses a basis formed of $n$ vector*

$$[\pi_{i1}], [\pi_{i2}], \dots, [\pi_{in}].$$

By putting

$$\lambda_i = \sum_{k=1}^{n} l_k \pi_{ik} \tag{11}$$

where $l_1, l_2, \dots, l_n$ are arbitrary integers, one will determine each vector $[\lambda_i]$ of the group $G$. By indicating

$$V(x_1, x_2, \dots, x_n, \pi_1, \pi_2, \dots, \pi_n) = p_{0k} + \sum_{i=1}^{n} p_i x_i, \quad (k = 1, 2, \dots, n)$$

and

$$a_i = \sum_{k=1}^{n} l_k p_{ik}, \tag{12}$$

one will have the formula

$$V(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_n) = \\ \sum_{k=1}^{n} l_k \left( p_{0k} - \frac{1}{2} \sum_{i=1}^{n} p_{ik} \pi_{ik} + \sum_{i=1}^{n} p_{ik} x_i \right) + \frac{1}{2} \sum_{i=1}^{n} a_i \lambda_i.$$

One will easily demonstrate Theorem II introduced with the help of the formula (4).

Let us notice that the sum $\sum a_i \lambda_i$ presents, by virtue of equation (11) and (12), a quadratic form of integral variables $l_1, l_2, \dots, l_n$

$$\sum a_i \lambda_i = \sum_{k=1}^{n} \sum_{h=1}^{n} A_{kh} l_k l_h,$$

where one has put

$$A_{kh} = \frac{1}{2} \sum_{i=1}^{n} p_{ik} \pi_{ih} + \frac{1}{2} \sum_{i=1}^{n} p_{ih} \pi_{ik}, \quad (k = 1, 2, \dots, n; h = 1, 2, \dots, n)$$

We will see that the quadratic form $\sum \sum A_{kh} l_k l_h$ obtained is positive.

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*Theorem I. Any primitive parallelohedron possesses a centre.*

Designate by $(\zeta_i)$ the point satisfying the equations

$$p_{0k} - \frac{1}{2}\sum_{i=1}^{n} p_{ik}\pi_{ik} + \sum_{i=1}^{n} p_{ik}\zeta_i = 0 \quad (k = 1, 2, \ldots, n) \tag{1}$$

I say that the point $(\zeta_i)$ represents the centre of the principal parallelohedron $R_0$

To demonstrate this, put

$$\lambda_{ih} = \sum_{k=1}^{n} l_k^{(h)} \pi_{ik} \quad (h = 1, 2, \ldots, \sigma)$$

By virtue of Theorem II of Number 40, one obtains

$$V(x_1, x_2, \ldots, x_n, \lambda_{1h}, \lambda_{2h}, \ldots, \lambda_{nh}) =$$

$$\sum l_k^{(h)} \left(p_{0k} - \frac{1}{2}\sum p_{ik}\pi_{ik} + \sum p_{ik}x_i\right) + \frac{1}{2}\sum_{i=1}^{n} \left(p_{i1}l_1^{(h)} + \ldots + p_{in}l_n^{(h)}\right) \lambda_{ih}$$

On the other hand, by virtue of the definition established in Number 32, one has

$$V(x_1, x_2, \ldots, x_n, \lambda_{1h}, \lambda_{2h}, \ldots, \lambda_{nh}) = a_{0h} + \sum a_{ih}x_i$$

It follows that

$$a_{ih} = \sum_{k=1}^{n} l_k^{(h)} p_{ik}, \quad (h = 1, 2, \ldots, \sigma) \tag{2}$$

and

$$a_{0h} = \sum_{k=1}^{n} l_k^{(h)} \left(p_0k - \frac{1}{2}\sum p_{ik}\pi_{ik}\right) + \frac{1}{2}\sum a_{ih}\lambda_{ih} \; (h = 1, 2, \ldots, \sigma) \tag{3}$$

Multiply the equation (1) by $(l^{(h)})$ and by attributing to the index $k$ the values $1, 2, \ldots, n$, add the equations obtained, it becomes, by (2) and (3),

$$a_{0h} - \frac{1}{2}\sum a_{ih}\lambda_{ih} + \sum a_{ih}\zeta_i = 0 \, (h = 1, 2, \ldots, \sigma) \tag{4}$$

That posed, take any one point $(x_i)$ belonging to the parallelohedron $R_0$ .

For the point $(\zeta_i)$ to be the centre of the parallelohedron $R_0$, it is necessary and sufficient that the point $(x'_i)$ determined by the equations

$$x'_i = 2\zeta_i - \lambda_i, \; (i = 1, 2, \ldots, n) \tag{5}$$

also belongs to the parallelohedron $R_0$ .

By virtue of the supposition made, one will have the inequalities

$$a_{0h} + \sum a_{ih} x_i \geq 0. \ (h = 1, 2, \ldots, \sigma) \qquad (6)$$

By noticing that by (4) and (5)

$$a_{0h} + \sum a_{ih} x'_i = -a_{0h} - \sum a_{ih}(x_i - \lambda_{ih})$$

and that the inequality

$$-a_{0h} - \sum a_{ih}(x_i - \lambda_{ih}) \geq 0$$

is found among the inequalities (6), one obtains

$$a_{0h} + \sum a_{ih} x'_i \geq 0. \quad (h = 1, 2, \ldots, \sigma)$$

It is therefore demonstrated that the point $(\xi_i)$ represents the centre of the parallelohedron $R_0$.

Let us notice that the centre $(\xi_i)$ is interior to the parallelohedron $R_0$.

To demonstrate this, let us suppose that a point $(x_i)$ is interior to the parallelohedron $R_0$.

One will have the inequalities

$$a_{0h} + \sum a_{ih} x_i > 0. \quad (h = 1, 2, \ldots, \sigma)$$

Among these inequalities can be found the inequalities

$$-a_{0h} - \sum a_{ih}(x_i - \lambda_{ih}) > 0. \quad (h = 1, 2, \ldots, \sigma)$$

By taking the summation of these inequalities, one obtains

$$\sum a_{ih} \lambda_{ih} > 0, \quad (h = 1, 2, \ldots, \sigma)$$

and, because of the equation (4), it becomes

$$a_{0h} + \sum a_{ih} \xi_i > 0. \quad (h = 1, 2, \ldots, \sigma)$$

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*Theorem II. The quadratic form*

$$\sum_{i=1}^{n} (p_{i1} l_1 + p_{i2} l_2 + \ldots + p_{in} l_n)(\pi_{i1} l_1 + \pi_{i2} l_2 + \ldots + \pi_{in} l_n)$$

Apply Theorem I of Number 37 to the centre $(\zeta_i)$ of the principal parallelohedron $R_0$, one will have the inequality

$$V(\zeta_1, \zeta_2, \ldots, \zeta_n, \lambda_1, \lambda_2, \ldots, \lambda_n) > 0, \tag{7}$$

whatever the vector $[\lambda_i]$ of the group $G$ may be, the vector $[0]$ being excluded.

By virtue of Theorem II of Number 40 and, [by virtue] of the equation (1), it becomes

$$p(\zeta_1, \zeta_2, \ldots, \zeta_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{2} \sum (p_{i1}l_1 + \ldots + p_{in}l_n)(\pi_{i1}l_1 + \ldots + \pi_{in}l_n)$$

and, from (7), one finds

$$\sum (p_{i1}l_1 + p_{i2}l_2 + \ldots + p_{in}l_n)(\pi_{i1}l_1 + \pi_{i2}l_2 + \ldots + \pi_{in}l_n) > 0$$

The inequality obtained holds, whatever the integral values of the variable $l_1, l_2, \ldots, l_n$ may be, the system $l_1 = 0, l_2 = 0, \ldots, l_n = 0$ being excluded.

### *Continuous group of the linear transformations of the primitive parallelohedra*

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Applying a linear transformation of the principal primitive parallelohedron $R_0$ with the help of a substitution

$$x_i = \alpha_{i0} + \sum_{k=1}^{n} \alpha_{ik} x'_k, \quad (i = 1, 2, \ldots, n)$$

with any real coefficients and of the determinant which does not vanish.

One obtains a new primitive parallelohedron $R'$ which will be determined with the help of the canonical inequalities

$$a'_{0h} + \sum_{k=1}^{n} a'_{kh} x'_k \geq 0, \quad (h = 1, 2, \ldots, \sigma)$$

where one has put

$$a'_{0h} = a_{0h} + \sum_{i=1}^{n} a_{ih}\alpha_{i0}, \; a'_{kh} = \sum_{i=1}^{n} a_{ih}\alpha_{ik} \tag{1}$$

$$(k = 1, 2, \ldots, h; h = 1, 2, \ldots, \sigma)$$

The group $G'$ of vectors corresponding to the parallelohedron $R'$ obtained will be determined by the equations

$$\lambda_i = \sum_{k=1}^{n} \alpha_{ik} \lambda_k, \tag{2}$$

on condition that the vector $[\lambda_i]$ of the group $G$ corresponds to the vector $[\lambda'_i]$ in the group $G'$ .

Designate

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = p_0 + \sum_{i=1}^{n} p_i x_i$$

and

$$V(x'_1, x'_2, \ldots x'_n, \lambda'_1, \lambda'_2, \ldots, \lambda'_n) = p'_0 + \sum_{i=1}^{n} p'_i x_i$$

By virtue of the formula (4) of Number 40 and [by virtue] of the equation (1) and (2) one obtains

$$p'_0 = p_0 + \sum_{i=1}^{n} p + i\alpha_{i0},\ p'_k = \sum_{i=1}^{n} p_i \alpha_{ik} \quad (k = 1, 2, \ldots, n)$$

Of which result $[\pi_i]$ and $[\pi'_i]$ being any two corresponding vectors, one will have

$$\sum_{i=1}^{n} p_i \pi_i = \sum_{i=1}^{n} p'_i \pi'_i \qquad (3).$$

*Theorem. The quadratic form*

$$\sum_{k=1}^{n} \sum_{h=1}^{n} A_{kh} l_k l_h = \sum_{i=1}^{n} (p_{i1} l_1 + p_{i2} l_2 + \ldots + p_{in} l_n)(\pi_{i1} l_1 + \pi_{i2} l_2 + \ldots + \pi_{in} l_n)$$

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Carry out a transformation of the primitive parallelohedra of the set $(R)$ with the help of a substitution

$$p_0 k - \frac{1}{2} \sum_{i=1}^{n} p_{ik} \pi_{ik} + \sum_{i=1}^{n} p_{ik} x_i = x'_k \quad (k = 1, 2, \ldots, n)$$

One obtains a set of the primitive parallelohedra $(R')$.

The corresponding value of the function $V(x'_1, x'_2, \ldots, x'_n, \lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ for the set $(R')$ will be expressed by the formular

$$V(x'_1, x'_2, \ldots, x'_n, \lambda'_1, \lambda'_2, \ldots, \lambda'_n) = \sum_{i=1}^{n} l_i x'_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} l_i l_j .$$

By virtue of the theorem Number 38, the principal parallelohedron of the set $(R)$ will be determined by the inequalities

$$\frac{1}{2}\sum\sum A_{ij}l_ih_j+\sum_{i=1}^{n}l_ix_i\geq 0$$

which hold, whatever the integral values of $l_1,l_2,\dots,l_n$ may be.

The various parallelohedra of the set $(R')$ will be determined by the inequalities

$$\frac{1}{2}\sum\sum A_{ij}l_il_j+\sum l_ix_i\geq\frac{1}{2}\sum\sum A_{ij}l_i^{(0)}l_j^{(0)}+\sum l_i^{(0)}x_i \qquad (4)$$

Each parallelohedron of the set $(R')$ will be characterised by a corresponding system $(l_i^{(0)})$ of integers $l_1^{(0)},l_2^{(0)},\dots,l_n^{(0)}$ .

Observe how one could replace the base of the group $G$ formed of $n$ vectors by another base also formed of $n$ vectors, these two bases will be equivalent, by virtue of Theorem III of Number 11; the corresponded positive quadratic form $\sum\sum A_{ij}l_il_j$ will be replaced by an equivalent form; the inequalities (4) define within this case the set of the parallelohedra which can be transformed as the set $(R')$ with the help of a corresponding linear substitution on integral coefficients and of the determinant $\pm 1$ .

The following remarkable theorem is thus demonstrated.

*Theorem. By applying the linear transformation of a primitive parallelohedron with the help of the substitutions in some real coefficients which form a group continuous for linear substitutions, one obtains a set of primitive parallelohedra which is perfectly determined by a class of equivalent positive quadratic form, on condition that one does not consider as being different the quadratic forms with proportional coefficients.*

We have seen how any positive quadratic form defines, by the help of the inequalities (4), a set of congruent parallelohedra which can be primitives or not.

## Section III

### Solution of the parallelohedra with the aid of positive quadratic form

*Definition of the convex polyhedron corresponding to a positive quadratic form*

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Let $\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j$ be an arbitrary positive quadratic form in $n$ variables $x_1,x_2,\dots,x_n$ . Imagine a set $R$ of points $(\alpha_i)$ satisfying the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j+2\sum_{i=1}^{n}\alpha_ix_i >\geq 0,$$

whatever may be the integral values of $x_1, x_2, \ldots, x_n$ .

By virtue of the definition established, the set $R$ enjoys the following properties:

1. The set $R$ is in $n$ dimensions
2. The point (0) represents the centre of the set $R$
3. The set $R$ is convex.

Take we a system of arbitrary parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ and examine a vector $g$ composed of points $(\alpha_i)$ which are determined by the equation

$$\alpha_i = \rho\epsilon_i \quad \text{where } \rho \geq 0$$

It is easy to demonstrate that there exists an interval

$$0 \leq \rho \leq \rho_0 \quad \text{where } \rho_0 > 0$$

which correspond with the points of vector $g$ belonging to the set $R$ .

By posing

$$\alpha_{i0} = \rho_0\epsilon_i,$$

one obtain a vector $[\alpha_{i0}]$ the points of which belong to the set $R$. The point $(\alpha_{i0})$ belongs to the boundary of the set $R$, that is to say: the point $(\alpha_{i0})$ satisfies the inequality

$$\sum\sum a_{ij}x_ix_j+2\sum\alpha_{i0}x_i \geq 0, \tag{1}$$

whatever may be the integral values of $x_1, x_2, \ldots, x_n$ and satisfy at least one equation

$$\sum\sum a_{ij}l_il_j+2\sum\alpha_{i0}l_i = 0, \tag{2}$$

$l_1, l_2, \ldots, l_n$ being the integers which do not vanish.

Designate

$$\alpha_{i1} = -\alpha_{i0}-\sum_{j=1}^{n}a_{ij}l_j, \quad (i = 1, 2, \ldots, n) \tag{3}$$

one will have, by (2), the equation

$$\sum\sum a_{ij}x_ix_j+2\sum\alpha_{i1}x_i = \sum\sum a_{ij}(l_i-x_i)(l_j-x_j)+2\sum\alpha_{i0}(l_i-x_i)$$

and, by virtue of (1), one obtains

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{i1}x_i \geq 0, \tag{4}$$

therefore the point $(\alpha_{i1})$ also belongs to the set $R$ .

By adding the inequalities (1) and (4), one finds, from (3),

$$\sum\sum a_{ij}x_ix_j - \sum\sum a_{ij}x_il_j \geq 0\,.$$

The inequality obtained holds, whatever the integral values of $x_1, x_2, \ldots, x_n$; this inequality can be written

$$\sum\sum a_{ij}l_il_j \leq \sum\sum a_{ij}(l_i - 2x_i)(l_j - 2x_j)$$

One concludes that the system $(l_i)$ is nothing but a representation of the minimum of the positive quadratic form $\sum\sum a_{ij}x_ix_j$ determined in the set composed of all the systems of integers which are contiguous to the system $l_i$ with respect to the modulo 2.

The number of such systems is finite. Suppose that all these systems form a series

$$(l_{i1}), (l_{i2}), \ldots, (l_{i\sigma}) \tag{5}$$

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*Theorem. The set R presents a convex polyhedron determined with the aid of the inequalities*

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_il_{ik} \geq 0\,(k = 1, 2, \ldots, \sigma) \tag{6}$$

By virtue of the definition established, each point $(\alpha_i)$ of the set $R$ satisfies these inequalities. Suppose that a point $(\alpha_i)$ satisfying these inequality does not belong to the set $R$. One will determine in this case a positive value of the parameter $\rho$ in the interval $0 < \rho < 1$, such that

$$\alpha_i^{(0)} = \rho\alpha_i \quad \text{where } 0 < \rho < 1, \tag{7}$$

one obtains a point $(\alpha_i^{(0)})$ belonging to the boundary of the set $R$. The point $(\alpha_i^{(0)})$ will satisfy, as we have seen, an equation

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i^{(0)}l_i = 0 \tag{8}$$

characterised by a system $(l_i)$ belonging to the series (6) .

By virtue of the equation obtained, one has

$$\sum \alpha_i^{(0)}l_i < 0$$

and, by (7), it becomes

$$\sum \alpha_i l_i < 0\,.$$

By presenting the equation (8) in the form

$$\sum\sum a_{ij} l_i l_j + 2\sum \alpha_i l_i = 2(1-\rho)\sum \alpha_i l_i,$$

one will have the inequality

$$\sum\sum a_{ij} l_i l_j + 2\sum \alpha_i l_i < 0,$$

which is contrary to the hypothesis.

### *Independent inequalities which define the convex polyhedron corresponding to a positive quadratic form*

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It may be the case that among the inequalities (6) of the previous number there are independent inequalities. Suppose, for example, that the inequality

$$\sum\sum a_i j l_i l_j + 2\sum \alpha_i l_i \geq 0 \tag{1}$$

ve dependent. One will have in this case an identity

$$\sum\sum a_i j l_i l_j + 2\sum \alpha_i l_i = \rho_0 + \sum_{k=1}^{\sigma} \rho_k \left(\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_i k\right) \tag{2}$$

where

$$\rho_0 \geq 0, \quad \rho_k \geq 0, \quad (k = 1, 2, \ldots, \sigma)\,.$$

We have seen in Numer 45 that the inequality

$$\sum\sum a_{ij} x_i x_j = \sum\sum a_{ij} x_i l_j \geq 0$$

holds whatever the integral values of $x_1, x_2, \ldots, x_m$ may be.

By making in the identity ($R$)

$$\alpha_i = -\frac{1}{2}\sum a_{ij} l_j,$$

one obtains

$$\rho_0 + \sum \rho_k \left(\sum\sum a_{ij} l_{ik} l_{jk} - \sum\sum a_{ij} l_{ik} l_j\right) = 0$$

and consequently

$$\rho_0 = 0,\ \rho_k \left(\sum\sum a_{ij} l_{ik} l_{jk} - \sum\sum a_{ij} l_{ik} l_j\right) = 0 \ (k = 1, 2, \ldots, \sigma)$$

By supposing that $\rho_k \neq 0$, one will have

$$\sum\sum a_{ij} l_{ik} l_{jk} - \sum\sum a_{ij} l_{ik} l_j = 0$$

thus

$$\sum\sum a_{ij} l_i l_j = \sum\sum a_{ij} (l_i - 2 l_i k)(l_j - 2 l_{jk})\,.$$

By virtue of the equation obtained, the system $(l_i - 2s_{ik})$ is in the series (5) of Number 45. This is a condition necessary for the inequality (1) to be dependent.

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*Theorem. For an inequality*

$$\sum\sum a_{ij} l_i l_j + 2\sum \alpha_i l_i \geq 0 \tag{3}$$

*to be independent, it is necessary and sufficient that the quadratic form* $\sum\sum a_{ij} x_i x_j$ *does not possess as two minimum representations* $(l_i)$ *and* $(-l_i)$ *in the set composed of all the systems of integers which are contiguous to the system* $(l_i)$ *with regard to the modulus 2.*

We have demonstrated that the condition studied is sufficient. It remains to be demonstrated that this condition is necessar.

Let us suppose that the inequality (3) is independent. In this case

$$\sum\sum a_{ij} l_i l_j + 2\sum \alpha_i l_i = 0$$

defines a face $P$ in $n - 1$ dimensions of the polyhedron $R$.

Let $(\alpha_i)$ be a point which is interior to the face $P$. One has the inequality

$$\sum\sum a_{ij} x_i x_j + 2\sum \alpha_i x_i > 0, \tag{4}$$

whatever the integral values of $x_1, x_2, \ldots, x_n$ may be, the two systems (0) and $(l_i)$ being excluded. By putting, as we have done in Number 45,

$$\alpha'_i = -\alpha_i - \sum a_{ij} l_j, \tag{5}$$

one will also have an inequality

$$\sum\sum a_{ij} x_i x_j + 2\sum \alpha'_i x_i > 0 \tag{6}$$

which holds for any integral values of $x_1, x_2, \ldots, x_n$, the two systems (0) and $(l_i)$ being excluded. By adding the inequality (4) and (6) one finds, by (5),

$$\sum\sum a_{ij} x_i x_j - \sum a_{ij} x_i l_j > 0$$

in other words

$$\sum\sum a_{ij}l_il_j < \sum\sum a_{ij}(l_i - 2x_i)(l_j - 2x_j).$$

The inequality obtained holds for any integral values of $x_1, x_2, \ldots, x_n$, the two systems $(0)$ and $(l_i)$ being excluded.

The theorem introduced is thus demonstrated.

*Corollary. The number of the independent inequalities which define the polyhedron $R$ corresponding to a positive quadratic form can not exceed the limit $2(2^n - 1)$.*

*Set $(R)$ of parallelohedra defined by a positive quadratic form.*

49

*Theorem. Let us suppose that the convex polyhedron $R$ corresponding to a positive quadratic form*

$\sum\sum a_{ij}x_ix_j$ *is determined with the help of the inequalities*

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_ix_i \geq 0.$$

*By applying the translations of polyhedron $R$ the length of the vector determined by the equations*

$$\lambda_i = -\sum a_{ij}l_j,$$

*$l_1, l_2, \ldots, l_n$ being the arbitrary integers, one will make up a set $(R)$ of congruent polyhedra which uniformly partition space in $n$ dimensions.*

Let us indicate with $R'$ the polyhedron which are obtained with the help of a translation of the polyhedron $R$ the length of the vector $[\lambda_i]$. The polyhedron $R'$ will be determined by the inequalities

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j + 2\sum_{i=1}^{n}(\alpha_i + \sum_{j=1}^{n} a_{ij}l_j)x_i \geq 0.$$

This inequality can be written

$$\sum\sum a_{ij}(x_i+l_i)(x_j+l_j)+2\sum \alpha_i(x_i+l_i) \geq \sum\sum a_{ij}l_il_j+2\sum \alpha_il_i.$$

One concludes that the polyhedron $R'$ will be determined by the inequalities

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_ix_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_il_i \qquad (1)$$

which hold, whatever the integral values of variables $x_1, x_2, \ldots, x_n$ may be.

One will say that the polyhedron $R'$ congruent with the polyhedron $R$ is characterised by the system $(l_i)$.

Let us indicate by $(R)$ the set of all the polyhedra congruent to polyhedron $R$ and which are characterised by the various systems $(l_i)$ of integers.

I argue that the set $(R)$ uniformly fills the space in $n$ dimensions.

Let us take an arbitrary point $(\alpha_i)$ in the space in $n$ dimensions and find the polyhedron of the set $(R)$ of which belongs the point $(\alpha_i)$. In this effect, determine a minimum representation $(l_i)$ of the form

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i$$

in the set $E$ composed of all the systems $(x_i)$ of integral values of the variables $x_1, x_2, \ldots, x_n$.

One will have the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i$$

which holds in the set $E$. As a result, the point $(\alpha_i)$ belongs to the polyhedron of the set $(R)$ characterised by the system $(l_i)$.

Let us suppose that the point $(\alpha_i)$ belongs to the various polyhedra of the set $(R)$: $R, R', \ldots, R^{(\mu)}$ characterised by the systems

$$(l_i), (l_{i1}), \ldots, (l_{i\mu}). \tag{2}$$

By virtue of (1), one obtains the inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i. \tag{3}$$
$$(k = 1, 2, \ldots, \mu)$$

It follows that one will have the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i > \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i,$$

for any integral values of $x_1, x_2, \ldots, x_n$, the systems (2) being excluded.

One concludes that the point $(\alpha_i)$ is interior to a face common to the polyhedra $R, R', \ldots, R^{(\mu)}$ and defined by the equations (3).

We have arrived at the following result: *Every positive quadratic form defines a set $(R)$ of congruent parallelohedra which can be primitive or not.*

*Algorithm for the search for the minimum of the form $\sum\sum a_{ij}x_ix_j +$*
50 *$2\sum \alpha_i x_i$ in the set $E$.*

Let us suppose that one had determined the independent inequalities

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{jk} \geq 0 \quad (k = 1, 2, \ldots, \sigma)$$

which define the parallelohedron $R$ corresponding to a positive quadratic form $\sum\sum a_{ij} x_i x_j$

With the help of the systems

$$(l_{i1}), (l_{i2}), \ldots, (l_{i\sigma})$$

of integers, one can resolve many problems of the arithmetic theory of positive quadratic form.

We seek, for example, the minimum of the form

$$\sum\sum a_{ij} x_i x_j = 2\sum a_i x_i \tag{1}$$

in the set $E$ composed of all the systems $(x_i)$ with integers, $\alpha_1, \alpha_2, \ldots,$ $\alpha_n$ being arbitrary parameters given.

The values of $x_1, x_2, \ldots, x_n$ which correspond to the absolute minimum of the function (1) verify the equations

$$\sum_{i=1}^{n} a_{ik} x_i + a_k = 0 \quad (k = 1, 2, \ldots, n)$$

We designate by $(\xi_i)$ the point verifying these equations. By posing

$$\xi_i = l_i + r_i$$

we determine the integers $l_1, l_2, \ldots, l_n$ under the conditions

$$|r_i| \leq \frac{1}{2} \qquad (i = 1, 2, \ldots, n)$$

In the case $r_i = 0 \, (i = 1, 2, \ldots, n)$ the system $(l_i)$ is the one we have sought. We suppose that all the numbers $r_i \, (i = 1, 2, \ldots, n)$ do not vanish. We pose

$$\alpha_i^{(0)} = \alpha_i + \sum_{j=1}^{n} a_{ij} l_j$$

and examine the point $(\alpha_i^{(0)})$ .

Let us suppose that the point $(\alpha_i)$ belongs to the parallelohedron of the set $(R)$ which is characterised by the system $(l_i)$, therefore the system represents the minimum of the form (1) .

In the case where the point $(\alpha_i^{(0)})$ does not belong to the parallelohedron $R$, we determine a value $\rho_0$ in the interval $0 < \rho_0 < 1$ of parameter $\rho$, in the manner such that the point $(\rho_0 \alpha_i^{(0)})$ belongs to a face of the parallelohedron $R$ . Suppose that this face be determined by the equation

$$\sum\sum a_{ij} l_{ih} l_{jh} + 2\sum \alpha_i l_{ih} = 0$$

One will have an equation

$$\sum\sum a_{ij} l_{ih} l_{jh} + 2\rho_0 \sum \alpha_i^0 l_{ih} = 0 \text{ where } 0 < \rho_0 < 1$$

Let

$$\alpha_i' = \alpha_i^0 + \sum_{j=1}^{n} a_{ij} l_{jh}$$

and examine anew the point $(\alpha_i')$ and so on. I say that one will always determine a representation of the minimum of the form (1) by repeating many times the procedure explained. To demonstrate, suppose that one had determined with the help of the algorithm shown a series of points

$$\left(\alpha_i^{(0)}\right), (\alpha_i'), \dots, \left(\alpha_i^{(k)}\right), \dots \tag{2}$$

and a series of systems

$$\left(l_i^{(0)}\right), (l_i'), \dots, \left(l_i^{(k)}\right), \dots ,$$

verifying the equations

$$\alpha_i^{(k)} = \alpha_i^{(k-1)} + \sum_{j=1}^{n} a_{ij} l_j^{(k-1)} \quad (k = 1, 2, \dots) \tag{3}$$

and the equations

$$\sum\sum a_{ij} l_i^{(k)} l_j^{(k)} + 2\sum \rho_k \alpha_i^{(k)} l_i^{(k)} = 0 \quad \text{where } 0 < \rho_k < 1 \quad (k = 0, 1, 2, \dots)$$

By virtue of these equations, one finds

$$\sum\sum a_{ij} l_i^{(k)} l_j^{(k)} + 2\sum \alpha_i^{(k)} l_i^{(k)} < 0 \quad (k = 0, 1, 2, \dots) \tag{4}$$

By designating

$$m_i^{(k)} = l_i + l_i^{(0)} + \cdots + l_i^{(k-1)} \quad (k = 1, 2, \dots) \tag{5}$$

and

$$m_i^{(0)} = l_i \, ,$$

one obtains, from (3),

$$\alpha_i^{(k)} = \alpha_i + \sum_{j=1}^{n} a_{ij} m_j^{(k)} \quad (k = 0, 1, 2, \ldots) \tag{6}$$

By substituting in the inequality (4), one gets

$$\sum\sum a_{ij} \left(l_i^{(k)} + m_i^{(k)}\right) \left(l_j^{(k)} + m_j^{(k)}\right) + 2\sum d_i \left(l_i^{(k)} + m_i^{(k)}\right) < \sum\sum a_{ij} m_i^{(k)} m_j^{(k)} + 2\sum \alpha_i m_i^{(k)}$$

This inequality, by (5), can be written

$$\sum\sum a_{ij} m_i^{k+1} m_j^{(}k+1) + 2\sum d_i m_i^{k+1} < \sum\sum a_{ij} m_i^{(k)} m_j^{(k)} + 2\sum d_i m_i^{(k)} \quad (k = 0, 1, 2, \ldots)$$

The number of the systems $(m_i^{(k)})$ of integers verifying these inequalities is limited. One concludes that the series of points (2) will always end by a point $(\alpha_i^{(k)})$ belonging to parallelohedron $R$ . By virtue of the equation (6), the system $(m_i^{(k)})$ represents the minimum of the form $\sum\sum a_{ij} x_i x_j + 2\sum\sum \alpha_i x_i$ in the set $E$. The problem described comes down to the search for all the parallelohedra of the set $(R)$ which are contiguous by a face in the interior of which the point $(\alpha_i^{(k)})$ is to be found. One will determine all these parallelohedra by successively determining the parallelohedra which are contiguous to $R$ through the faces in $n-1$ dimensions and so on and so forth.

### *Properties of the systems of integers which characterise the faces in $n-1$ dimensions of the parallelohedron corresponding to a positive quadratic form*

51 Suppose that the systems

$$\pm(l_{i1}), \pm(l_{i2}), \ldots, \pm(l_{iT}) \tag{1}$$

characterises the faces in $n-1$ dimensions of parallelohedron $R$ corresponding to a positive quadratic form

$$\sum\sum a_{ij} x_i x_j$$

*Theorem I. The elements $l_1k, l_2k, \ldots, l_mk$ of any system $(l_{ik})$ belonging to the series (1) have no common divisor.*

We have seen in Number 45 that the numbers $l_1k, l_2k, \ldots, l_nk$ verify the inequality

$$\sum\sum a_{ij}x_ix_j - \sum\sum a_{ij}x_il_{jk} \geq 0$$

in the set $E$. By letting

$$l_{ik} = \delta t_i \quad \text{where } \delta \geq 1$$

and by putting $x_i = t_i$ in the previous inequality, one gets

$$\sum\sum a_{ij}t_it_j - \delta\sum\sum a_{ij}t_it_j \geq 0$$

and it is necessary that $\delta = 1$.

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*Theorem II. Suppose that n systems*

$$(p_{i1}), (p_{i2}), \ldots, (p_{in}) \tag{2}$$

*represent n consecutive minima*

$$M_1 \leq M_2 \leq \ldots M_n$$

*of the positive quadratic form* $\sum\sum a_{ij}x_ix_j$. *All the systems (2) are in the series (1).*

By virtue of the definition for the system of $n$ consecutive minima, one will have an inequality

$$M_k = \sum\sum a_{ij}p_{ik}p_{jk} \leq \sum\sum a_{ij}x_ix_j \quad (k = 1, 2, \ldots, n)$$

as long as all the numbers $x_1, x_2, \ldots, x_n$ can not be presented in the form

$$M_k = \sum_{r=1}^{k-1} u_rp_ir\,,$$

the system (0) being excluded.

Suppose that the system $(p_{ik})$ does not belong to the series (1). In this case there exists a system $(t_i)$ of all the numbers verifying the inequality

$$\sum\sum a_{ij}p_{ik}p_{jk} \geq \sum\sum a_{ij}(p_{ik} - 2t_i)(p_{jk} - 2t_j)$$

On letting

$$q_i = p_{ik} - t_i \tag{3}$$

one presents the previous inequality in the form

$$\sum\sum a_{ij}t_it_j + \sum\sum a_{ij}q_iq_j \qquad (4)\,.$$

By supposing that the two systems $(t_i)$ and $(q_i)$ are different from the system (0), one will have, by virtue of the inequality obtained, the equation

$$t_i = \sum_{r=1}^{k-1} u_r p_i r\,, \quad q_i = \sum_{r=1}^{k-1} v_r p_i r$$

and, from (3), it follows that

$$p_{ik} = \sum_{r=1}^{k-1} (u_r + v_r) p_i r\,.$$

The equations obtained are impossible, since otherwise the determinant of $n$ systems (2) would vanish, which is contrary to the hypothesis.

As a result, the inequality (4) does not hold at condition where either

$$t_i = 0 \quad \text{or} \quad t_i = p_{ik} \quad (i = 1, 2, \dots, n)$$

It is therefore demonstrated that the system $(p_{ik})$, $(k = 1, 2, \dots, n)$ belongs to the series (1).

*Corollary. All the representations for the arithmetic minimum of the positive quadratic form* $\sum\sum a_{ij}x_ix_j$ are within the series (1).

53

*Theorem III. The numerical value of determinant of any $n$ systems which belong to the series (1) is less than $n$!*

Choose any $n$ systems in the series (1)

$$(l_{i1}), (l_{i2}), \dots, (l_{in})$$

which the determinant $\pm\omega$ does not vanish. Let us indicate

$$\alpha_{ik}^{(0)} = \frac{1}{2}\sum_{j=1}^{n} a_{ij} l_{jk} \text{ and } \alpha'_{ik} = -\frac{1}{2}\sum_{j=1}^{n} a_{ij} l_{jk} \ (k = 1, 2, \dots, n) \tag{5}$$

By virtue of the inequalities

$$\sum\sum a_{ij}x_ix_j \pm \sum\sum a_{ij}x_il_{jh} \geq 0 \quad (h = 1, 2, \dots, \tau)$$

which holds in the set $E$, $2n$ points (5) that belong to the parallelohedron $R$ corresponding to the quadratic form $\sum\sum a_{ij}x_ix_j$.

Let us choose any $n$ points among $2n$ points (5), making sure that two points corresponding to the same index $k$ value are not among the ones chosen. One forms in this manner $2n$ systems composed of $n$ points

$$\left(\alpha_{ih_1}^{(0)}\right), \left(\alpha_{ih_2}^{(0)}\right), \dots, \left(\alpha_{ih_\mu}^{(0)}\right), (\alpha'_{ih_{\mu+1}}), \dots, (\alpha'_{ih_n})$$

$h_1, h_2, \dots, h_n$ being any permutation of the indices $1, 2, \dots, n$ and $\mu = 0, 1, 2, \dots, n$ .

We designate, to summarise,

$$\alpha_{ik}^{(h)} = \alpha_{ih_k}^{(0)}, \ (k = 1, 2, \dots, \mu) \ \alpha_{ik}^{(h)} = \alpha'_{ih_k}, \qquad (6)$$

$$(k = \mu + 1, \dots, n; h = 1, 2, \dots, 2^n)$$

and examine a simplex $K_h$ determined by the equation

$$x_i = \sum_{k=1}^{n} \vartheta_k \alpha_{ik}^{(h)} \text{ where } \sum_{k=1}^{n} \vartheta_k \leq 1 \text{ and } \vartheta_k \geq 0 \ (k = 1, 2, \dots, n)$$

All the simplexes $K_h$, $(h = 1, 2, \dots, 2^n)$ belong to the parallelohedron $R$ . Any point $(\alpha_i)$, which is interior to a simplex $K_h$, does not belong to any other simplex of the series formed. This results in an inequality

$$\sum_h \int_{(K_h)} dx_1 dx_2 \cdots dx_n < \int_{(R)} dx_1 dx_2 \cdots dx_n \quad (h = 1, 2, \dots, 2^n) \ (7)$$

54 On designating by $D$ the determinant

$$D = \begin{vmatrix} a_1 1 & a_1 2 & \cdots & a_1 n \\ \vdots & \vdots & \ddots & \vdots \\ a_n 1 & a_n 2 & \cdots & a_n n \end{vmatrix}$$

of the quadratic form $\sum \sum a_{ij} x_i x_j$, one has by virtue of (5) and (6)

$$\int_{(K_h)} dx_1 dx_2 \cdots dx_n = \frac{\omega}{n!} \cdot \frac{D}{2^n}$$

and the inequality (7) gives

$$\frac{\omega}{n!} D < \int_{(R)} dx_1 dx_2 \cdots dx_n \qquad (8)$$

This established, we observe how the group $G$ of vectors corresponding to the parallelohedron $R$ possesses a basis formed by $n$ vectors

$$[a_{i1}], [a_{i2}], \dots, [a_{in}] .$$

By virtue of Theorem III of Number 11, it follows that

$$\int_{(R)} dx_1 dx_2 \cdots dx_n = D\,. \qquad (9)$$

By substituting in the inequality (8), one would obtain

$$\omega < n!$$

New application of continuous parameters to the theory of quadratic form.

## Second Memoir

Studies on the primitive parallelohedra
by Mr. Georges Voronoi in Warsaw

## Second Part

Domains of quadratic forms
corresponding to the various types of primitive parallelohedra

## Section IV

Various types of primitive parallelohedra

[*Journal für die reine und angewandte Mathematik*, V. 136, p. 67–181, 1909]
[translated by K N Tiyapan]

*On the number of faces in $n-1$ dimensions of primitive parallelohedron.*

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*Theorem. The number of faces in $n-1$ dimensions of a primitive parallelohedron is equal to* $2(2^n-1)$.

Let us suppose that a primitive parallelohedron $R$ corresponding to a positive quadratic form $\sum\sum a_{ij}x_ix_j$ is defined by the independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} \geq 0. \ (k=1,2,\ldots,\tau) \tag{1}$$

We have seen in Number 48 that any system

$$\pm(l_{1k}, l_{2k}, \ldots, l_{nk}), \ (k=1,2,\ldots,\tau) \tag{2}$$

represents the minimum of the quadratic form $\sum\sum a_{ij}x_ix_j$ in the set composed of all the systems of integers which are congruent to the system $\pm(l_{ik})$ with respect to the modulus 2. The form $\sum\sum a_{ij}x_ix_j$ possesses in this set only two minimum representations $\pm(l_{ik})$.

Let us divide the set $E$, composed of all the systems $(x_i)$ of integral numbers $x_1, x_2, \ldots, x_n$, into $2^n$ classes

$$E_0, E_1, \ldots, E_m \text{ where } m = 2^n-1$$

with regard to the modulus 2 and suppose that the set $E_0$ is composed of systems the elements of which have the common divisor 2.

All the systems (2) do not belong to the different sets

$$E_1, E_2, \ldots, E_m \text{ where } m = 2^n-1.$$

It follows that

$$\tau \leq 2^n - 1.$$

I argue that $\tau = 2^n - 1$. Let us suppose that among the systems (2) there are not found the systems belonging to a set $E$ and we determine the minimum of the form $\sum\sum a_{ij}x_ix_j$ in the set $E_h$. Let $(l_i)$ be a representation of this minimum.

Let us indicate by

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is}) \tag{3}$$

the vertices of the parallellohedron $R$ defined by the inequalities (1) and examine the values of the function $\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i$ which correspond to the different vertices (3). Let us suppose that the sum $\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i$ be the smallest one.

By virtue of the supposition made, one will have the inequalities

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ir}l_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i. \, (r = 1, 2, \ldots, s) \tag{4}$$

By noticing that each point $(\alpha_i)$ belonging to the parallelohedron $R$ can be determined by the equalities

$$\alpha_i = \sum_{r=1}^{s} \vartheta_r \alpha_{ir} \text{ where } \sum \vartheta_r = 1 \, \text{and} \, \vartheta_r \geq 0, (r = 1, 2, \ldots, s)$$

one will deduce the inequalities (4) an inequality

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i$$

which holds for any poit $(\alpha_i)$ belonging to the parallelohedron $R$. The system $(l_i)$ which represents the minimum of the form $\sum\sum a_{ij}x_ix_j$ in the set $E_h$ verifies the inequality

$$\sum\sum a_{ij}x_ix_j - \sum\sum a_{ij}x_il_j \geq 0$$

in the set $E$. It results that the point

$$\xi_i = -\frac{1}{2}\sum_{j=1}^{n} \alpha_{ij}l_j$$

belong to the parallelohedron $R$. By making in the inequality (5) $\alpha_i = \xi_i$, one notices

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i \leq 0.$$

The vertice $(\alpha_{ik})$ of the parallelohedron $R$ verifies the inequality

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i \geq 0,$$

therefore it is necessary that

$$\sum\sum a_{ij} l_i l_j + 2\sum \alpha_{ik} l_i = 0. \tag{6}$$

This stated, let us notice that the vertice $(\alpha_i k)$ of the primitive parallelohedron is simple.

Let us indicate by

$$\sum\sum a_{ij} l_{ir}^{(k)} l_{jr}^{(k)} + 2\sum \alpha_{ik} l_{ir}^{(k)} = 0, \quad (r = 1, 2, \ldots, n)$$

$n$ equations which define the vertice in the parallelohedron $R$. As the vertice $(\alpha_{ik})$ is simple, one will have an inequality

$$\sum\sum a_{ij} x_i x_j + 2\sum \alpha_{ik} x_i > 0,$$

whatever the integral values of $x_1, x_2, \ldots, x_n$ may be, the system (0) and the systems

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{in}^{(k)}) \tag{7}$$

being excluded. By virtue of the equality (6), the system $(l_i)$ is found among the systems (7) which all belong to the series (R).

It is therefore demonstrated that

$$\tau = 2^n - 1$$

and that the number of faces in $n-1$ dimensions of the parallelohedron $R$ is equal to

$$2\tau = 2(2^n - 1).$$

*Definition of the type of primitive parallelohedra.*

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Let us examine a primitive parallelohedron $R$ determined with the help of independent inequalities

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} \geq 0. \ (k = 1, 2, \ldots, \sigma \text{ where } \sigma = 2(2^n - 1))$$

Let us indicate with

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

the vertices of the parallelohedron $R$. One will determine with the help of equations

$$\sum\sum a_{ij} l_{ir}^{(k)} l_{jr}^{(k)} + 2\sum \alpha_{ik} l_{ir}^{(k)} = 0. \quad (r = 1, 2, \ldots, n, k = 1, 2, \ldots, s)$$

Each vertice $(\alpha_{ik})$ $(k = 1, 2, \ldots, s)$ is characterised by $n$ systems of integers

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{in}^{(k)}), \quad (k = 1, 2, \ldots, s) \tag{1}$$

the determinant $\pm\omega_k$ of which does not cancel each other out.

Let us indicate, to make short, $n$ systems (1) by a symbol

$$\{l_{ir}^{(k)}\}.$$

All the vertices of the primitive parallelohedron $R$ will be characterised by a set of symbols

$$\{l_{ir}^{(1)}\}, \{l_{ir}^{(2)}\}, \ldots, \{l_{ir}^{(s)}\}. \tag{2}$$

This declared, let us examine another primitive parallelohedron $R'$ corresponding to another positive quadratic form $\sum\sum a'_{ij}x_ix_j$. It can turn out that all the vertices of the parallelohedron $R'$ will alsobe characterised by the symbols (2). One will say in this case that the two parallelohedra $R$ and $R'$ belong to the same type.

*Definition. One will call the various parallelohedra all the vertices of which are characterised by the set of symbols (2), "belonging to the same type."*

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One can characterise a type of primitive parallelohedra in many ways.

Let us consider a set $(R)$ of congruent primitive parallelohedra which corresponds to a positive quadratic form $\sum\sum a_{ij}x_ix_j$.

All the vertices of parallelohedra belonging to the set $(R)$ can be divided into classes of congruent vertices. Let us indicate by $\tau$ the number of incongruent vertices belonging to the various classes.

Any vertice of a primitive parallelohedron is congruent to $n$ vertices of parallelohedron, this results in that

$$S = (n+1)\tau.$$

Let $(\alpha_i)$ be any one vertice of parallelohedra of the set $(R)$. One will define it with the help of $n+1$ equations

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = A. \ (k = 0, 1, 2, \ldots, n) \qquad 3$$

The $n+1$ systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in})$$

characterise $n+1$ parallelohedra of the set $(R)$ which are contiguous by the vertice $(\alpha_i)$. By indicating with $(l_i)$ a system of arbitrary integers, one will characterise by the systems

$$(l_{i0}+l_i), (l_{i1}+l_i), \dots, (l_{in}+l_i) \tag{4}$$

all the congruent vertices of parallelohedra of the set $(R)$.

By attributing to the variables $l_1, l_2, \dots, l_n$ any arbitrary values, one will characterise by $n+1$ systems (4) a class of congruent vertices.

One concludes this that a type of primitive parallelohedra can be characterised by $\tau$ systems (4).

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To have more convenience in the notations, let us introduce in our studies the linear functions

$$u = \sum_{i=1}^{n} l_i x_i, \quad \text{and} \quad u_k = \sum_{i=1}^{n} l_{ik} x_i. \quad (k = 0, 1, 2, \dots, n)$$

One will say that the symbol $(u_0, u_1, \dots, u_n)$ characterise the vertice $(\alpha_i)$ determined by the equations (3); the symbol $(u_0+u, u_1+u, \dots, u_n+u)$, $u$ being a linear function in arbitrary integral coefficients, characterise a vertice congruent to the vertice $(\alpha_i)$.

Let us suppose that one had characterised by the symbols

$$(u_0^{(k)}, u_1^{(k)}, \dots, u_n^{(k)}), \quad (k = 1, 2, \dots, \tau) \tag{5}$$

$\tau$ congruent vertices of primitive parallelohedra belonging to the set $(R)$. One will say that the set of symbols (5) characterise a type of primitive parallelohedra.

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Let us examine the faces in various dimensions of primitive parallelohedra belonging to the same type.

Let $P(\nu)$ be a face in $\nu$ dimensions $(\nu = 0, 1, 2, \dots, n-1)$ of parallelohedra of the set $(R)$ defined by the equations

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = \sum\sum a_{ij} l_{i0} + 2\sum \alpha_i l_{i0}. (k = 1, 2, \dots, n-\nu)$$

One will characterise this face by $n+1-\nu$ systems

$$(l_{i0}), (l_{i1}), \dots, (l_{i,n-\nu})$$

or by $n+1-\nu$ corresponding linear functions.

$$u_0, u_1, \dots, u_{n-\nu}.$$

All the faces in $\nu$ dimensions of parallelohedra of the set $(R)$ which are congruent to the face $P(\nu)$ will be characterised by the systems

$$(l_{i0}l_i), (l_{i1}l_i), \ldots, (l_{i,n-\nu}l_i)$$

or by the corresponding linear functions

$$u_0 + u, u_1 + u, \ldots, u_{n-\nu} + u.$$

By making, for example, $l_i = -l_{i0}$ one obtains $n - \nu$ systems

$$(l_{i1} - l_{i0}), (l_{i2} - l_{i0}), \ldots, (l_{i,n-\nu} - l_{i0}) \qquad (6)$$

which enjoy the following property: all the determinants of the order $(n - \nu)^2$ which one can form from $n - \nu$ systems (6) do not cancel one another at the same time. Let us indicate by $\omega^{(n-\nu)}$ the greatest common divisor of these determinants. By declaring

$$x_i = \sum_{k=1}^{n-\nu}(l_{ik} - l_{i0})\xi_k, \qquad (7)$$

one will present a system $(x_i)$ of integers by the linear forms where $\xi_1, \xi_2, \ldots, \xi_{n-\nu}$ are integral or rational numbers which belong to $\omega^{(n-\nu)}$ sets

$$\xi_k = \vartheta_{kr} + y_k \quad (k = 1, 2, \ldots, n - \nu; r = 1, 2, \ldots, \omega^{(n-\nu)}) \qquad (8)$$

where $y_1, y_2, \ldots, y_{n-\nu}$ are arbitrary integers. Among the sets (8) is found a set where $\vartheta_{kr} = 0,\ k = 1, 2, \ldots, n - \nu$ and which is composed of integral values of $\xi_1, \xi_2, \ldots, \xi_{n-\nu}$.

In the case $\omega^{(n-\nu)} = 1$, the equalities (7) are possible only on condition that the number $\xi_1, \xi_2, \ldots, \xi_{n-\nu}$ be integral.

The set (8) play an important role in the subsequent studies.

Let us indicate by the symbol $\sigma_{n-\nu}$ the number of incongruent faces in $\nu$ dimensions of primitive parallelohedra belonging to the type examined. By indicating with the symbol $S_\nu$ the number of faces in $\nu$ dimensions of corresponding primitive parallelohedron, one will have a formula

$$S_\nu = (n + 1 - \nu)\sigma_{n-\nu}. \quad (\nu = 0, 1, 2, \ldots, n - 1) \qquad (9)$$

*Definition of the set $(L)$ of simplexes characterising a type of primitive parallelohedra*

Let us suppose that $n+1$ systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \tag{1}$$

characterise a vertice of primitive parallelohedra belonging to the type examined.

*Definition I. One will call correlative to the vertice of primitive parallelohedra characterised by the systems* (1) *a simplex* $L$ *having* $n+1$ *vertices*

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}).$$

The simplex $L$ presents a set of points determined by the equalities

$$x_i = \sum_{k=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0. \quad (k = 0, 1, 2, \ldots, n)$$

Let us indicate by $(L)$ the set of simplexes correlative to the various vertices of a set $(R)$ of primitive parallelohedra belonging to the type examined.

*Definition II. One will say that a type of primitive parallelohedra is characterised by the set* $(L)$ *of simplexes.*

One will call congruent two simplexes characterised by the vertices

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \text{ and } (l_{i0}+l_i), (l_{i1}+l_i), \ldots, (l_{in}+l_i),$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers.

All the simplexes of the set $(L)$ can be divided into classes of congruent simplexes; the number of classes is expressed by the symbol $\sigma_n$ defined by the formula (9) of the previous number.

With the help of equations

$$x_i = \sum_{k=0}^{n-\nu} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n-\nu} \text{ and } \vartheta_k \geq 0, \quad (k = 0, 1, 2, \ldots, n-\nu)$$

one will determine a face in $n-\nu$ dimensions of the simplex $L$ which is correlative to the face in $\nu$ dimensions of parallelohedra characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{i[,]n-\nu}).$$

One concludes that the number of incongruent faces in $n-\nu$ dimensions of the set $(L)$ of simplexes is expressed by the symbol $\sigma_{n-\nu}$ $(\nu = 0, 1, 2, \ldots, n-1)$.

As all the vertices of simplexes of the set $(L)$ are congruent, one will declare $\sigma_0 = 1$, and the formula (9) of Number 59

$$S_\nu = (n+1-\nu)\sigma_{n-\nu}$$

will hold for the values of $\nu = 0, 1, 2, \ldots, n$, provided that one would admit $S_n = 1$.

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*Theorem I. The set $(L)$ of simplexes uniformly fills the space in $n$ dimensions.*

Let us suppose that a point $(x_i)$ be interior to a face of the simplex $L$ characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{i\nu}). \tag{2}$$

One will have

$$x_i = \sum_{k=0}^{\nu} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{\nu} \vartheta_k = 1 \text{ and } \vartheta_k > 0. \ (k = 0, 1, 2, \ldots, \nu) \tag{3}$$

Let us suppose that the point $(x_i)$ be interior to a face in $\nu'$ dimensions of another simplex $L'$ characterised by the vertices

$$(l'_{i0}), (l'_{i1}), \ldots, (l'_{i\nu}).$$

One can write

$$x_i = \sum_{k=0}^{\nu'} \vartheta'_k l'_{ik} \text{ where } \sum_{k=0}^{\nu'} \vartheta'_k = 1 \text{ and } \vartheta'_k > 0. \, a; (k = 0, 1, 2, \ldots, \nu') \tag{4}$$

Let $\sum\sum a_{ij} x_i x_j$ be a positive quadratic form which defines a set $(R)$ of primitive parallelohedra belonging to the type examined.

Let us indicate by $(\alpha_i)$ and $(\alpha'_i)$ two vertices of parallelohedra of the set $(R)$ which are correlative to the simplexes $L$ and $L'$. One will have the equalities

$$\left\{ \begin{array}{l} \sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = A, \\ \sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha'_i l'_{ik} = A'. \end{array} \right. \quad (k = 0, 1, 2, \ldots, n) \tag{5}$$

By putting down

$$\left\{ \begin{array}{l} \sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_i l'_{ik} = A + \rho_k, \\ \sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha'_i l_{ik} = A' + \rho'_k, \end{array} \right. \quad (k = 0, 1, 2, \ldots, n) \tag{6}$$

one will have the inequalities

$$\rho_k \geq 0 \text{ and } \rho'_k \geq 0. \quad (k = 0, 1, 2, \ldots, n)$$

From equalities (5) and (6) one derives

$$\begin{cases} A' - A + 2\sum(\alpha_i - \alpha'_i)l'_{ik} = \rho_k, \\ A - A' + 2\sum(\alpha'_i - \alpha_i)l_{ik} = \rho'_k. \end{cases} \quad (k = 0, 1, 2, \ldots, n)$$

By virtue of equalities (3) and (4), one obtains,

$$A' - A + 2\sum(\alpha_i - \alpha'_i)x_i = \sum_{k=0}^{\nu'} \rho_k \vartheta'_k,$$

$$A - A' + 2\sum(\alpha'_i - \alpha_i)x_i = \sum_{i=0}^{\nu} \rho'_k \vartheta_k.$$

By making the sum of these equalities, one finds

$$\sum_{k=0}^{\nu'} \rho_k \vartheta'_k + \sum_{k=0}^{\nu} \rho'_k \vartheta_k = 0.$$

It follows, because of (3) and (4), that

$$\rho_k = 0, \quad (k = 0, 1, 2, \ldots, \nu') \quad \text{and} \quad \rho'_k = 0. \quad (k = 0, 1, 2, \ldots, nu)$$

let us notice that the equality $\rho_k = 0$ is possible only on the condition that the system $(l'_{ik})$ is found among the vertices of the simplex $L$, similarly, the equality $\rho'_k = 0$ is possible only on condition that the system $(l_{ik})$ is found among the vertices of the simplex $L'$.

One concludes that the systems

$$(l'_{i0}), (l'_{i1}), \ldots, (l'_{i\nu'}) \tag{7}$$

characterise a face of the simplex $L$ and that the systems (2) characterise a face of the simplex $L'$. As a point $(x_i)$ can not be interior to two different faces of the same simplex, it results in that the systems (2) and (7) coincide; therefore the two simplexes $L$ and $L'$ are contiguous through the faces in $\nu$ dimensions characterised by the systems (2).

It remains to demonstrate that any point $(x_i)$ of the space in $n$ dimensions belongs to at least one simplex of the set $(L)$.

To demonstrate this, let us take any one point $(\xi_i)$ which is interior to the simplex $L$ and draw any one curve $C$ which joins the points $(\xi_i)$ and $(x_i)$. I say that all the points of the curve $C$ will be situated in the simplexes

$$L, L', \dots, L^{(m)}$$

belonging to the set $(L)$. In effect, let us suppose that the point $(x_i)$ not belong to the simplex $L$. The curve $C$ will go beyond in one point $(\xi_i')$ the boundary of the simplex $L$ and will pass through a simplex $L'$ which is contiguous to the simplex $L$ through a face in any one number of dimensions and so on and so forth.

62 *Theorem II. A point $(x_i)$ the elements $x_1, x_2, \dots, x_n$ of which are integers can be only one vertex of simplexes of the set $(L)$.*

Let us notice that there exist simplexes of the set $(L)$ which passes the vertex $(0)$; the number of these simplexes is expressed by the symbol $S_0$.

By effecting the translations of these simplexes the length of the vector $[x_i]$, one will obtain $S_0$ simplexes which possess the vertice $(x_i)$. By virtue of Theorem I, the point $(x_i)$ can not belong to other simplexes of the set $(L)$.

*Corollary. Suppose that a point $(x_i)$ the elements $x_1, x_2, \dots, x_n$ of which are integers, is not found among the vertices*

$$(l_{i0}), (l_{i1}), \dots, (l_{in})$$

*of a simplex $L$. By writing*

$$x_i = \sum_{i=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1,$$

*one will have among the numbers $\vartheta_0, \vartheta_1, \dots, \vartheta_n$ at least one negative number.*

*Properties of symbols $S_\nu$ and $\sigma_\nu$ $(\nu = 0, 1, 2, \dots, n)$.*

63 Let us take any one positive integer $m$ and consider a set $K$ of points which are congruent to $m^n$ points

$$\frac{g_1}{m}, \frac{g_2}{m}, \dots, \frac{g_n}{m}$$

which one obtains by attributing to the numbers $g_1, g_2, \dots, g_n$ the integral values verifying the inequalities

$$0 \le g_k < m. \quad (k = 1, 2, \dots, n)$$

Let us take any one point $(\frac{x_i}{m})$ of the set $K$ and suppose that the point $(\frac{x_i}{m})$ be interior to any one face $P(\nu)$ of simplexes of the set $(L)$ $(\nu = 0, 1, 2, \dots, n)$. By virtue of Theorem I of number 61, all points of the set $K$ which are interior to the face $P(\nu)$ can not be congruent.

Let us indicate by

$$P_k^{(\nu)}, \quad (k = 1, 2, \dots, \sigma_\nu; \nu = 0, 1, 2, \dots, n)$$

the various incongruent faces of simplexes of the set $(L)$ and by the symbol

$$m_k^{(\nu)} \quad (k = 1, 2, \dots, \sigma_\nu; \nu = 0, 1, 2, \dots, n)$$

let us indicate the number of points of the set $K$ which are interior to the face $P_k(\nu)$. ONe will have a formula

$$\sum_{\nu=0}^{n} \sum_{k=1}^{\sigma_\nu} m_k^{(\nu)} = m^n. \tag{1}$$

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It is easy to determine the value of the symbol $m_k^{(\nu)}$. Let us indicate by

$$l_{i0}^{(k)}, l_{i1}^{(k)}, \dots, l_{i\nu}^{(k)}$$

the vertices of the face $P_k(\nu)$ and let us write

$$\frac{x_i}{m} = \sum_{r=0}^{\nu} \vartheta_r l_{ir}^{(k)} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r > 0. \quad (r = 0, 1, 2, \dots, \nu)$$

These equalities can be written

$$\frac{x_i}{m} - l_{i0}^{(k)} = \sum_{r=1}^{\nu} \vartheta_r (l_{ir}^{(k)} - l_{i0}^{(k)}).$$

By indicating, to make short,

$$x_i - m l_{i0}^{(k)} = t_i, \;\; l_{ir}^{(k)} - l_{i0}^{(k)} = p_{ir}, \quad (r = 1, 2, \dots, \nu)$$

and

$$m\vartheta_r = \tau_r, \quad (r = 1, 2, \dots, \nu)$$

one will have

$$t_i = \sum_{r=1}^{\nu} \tau_r p_{ir} \text{ where } \sum_{r=1}^{\nu} \tau_r < m \text{ and } \tau_r > 0. \; (r = 1, 2, \dots, \nu) \tag{2}$$

Let us indicate by $\omega_k^{(\nu)}$ the greatest common divisor of determinants of the order $\nu^2$ which one can form from $\nu$ systems

$$(p_{i1}), (p_{i2}), \dots, (p_{i\nu})$$

and suppose that the forms (2) represent the integers $t_1, t_2, \dots, t_n$, provided that the numbers $\tau_1, \tau_2, \dots, \tau_\nu$ belong to one of $\omega_k^{(\nu)}$ sets

$$\tau_r = \xi_{rh} + y_r \text{ where } r = 1, 2, \dots, \nu, h = 1, 2, \dots, \omega_k^{(\nu)}, \tag{3}$$

$y_1, y_2, \dots, y_\nu$ being arbitrary integers.

One can suppose that

$$0 < \xi_{rh} \leq 1. \quad (r = 1, 2, \dots, \nu, h = 1, 2, \dots, \omega_k^{(\nu)})$$

By substituting the expressions of $\tau_r$ $(r = 1, 2, \dots, \nu)$ derived from equalities (3) in the inequalities (2), one obtains

$$\xi_{rh} + y_r > 0, (r = 1, 2, \dots, \nu) \quad \sum_{r=1}^{\nu} (\xi_{rh} + y_r) < m. \tag{4}$$

Let us indicate

$$\sum_{r=1}^{\nu} \xi_{rh} = a_h + \xi_h \tag{5}$$

where the integer $a_h$ is determined after the conditions

$$0 \leq \xi_p < 1. \tag{6}$$

The inequalities (4) can be replaced by the following ones:

$$y_r \geq 0, \ (r = 1, 2, \dots, \nu) \text{ and } \sum_{r=1}^{\nu} y_r \leq m - a_h - 1.$$

The number of systems $(y_1, y_2, \dots, y_\nu)$ of integers $y_1, y_2, \dots, y_\nu$ verifying these inequalities is equal to

$$\frac{(m - a_h)(m + 1 - a_h) \cdots (m + \nu - 1 - a_h)}{1 \cdot 2 \cdots \nu.}$$

By replacing with $a_{hk}^{(\nu)}$ the number $a_h$ corresponding to the various sets (3), one obtains the formula

$$m_k^{(\nu)} = \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m - a_{hk}^{(\nu)})(m + 1 - a_{hk}^{(\nu)}) \cdots (m + \nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu}. \tag{7}$$

By substituting in the equality (1), one finds

$$\sum_{\nu=0}^{n}\sum_{k=1}^{\sigma_\nu}\sum_{h=1}^{\omega_k^{(\nu)}}\frac{(m-a_{hk}^{(\nu)})\cdots(m+\nu-1-a_{hk}^{(\nu)})}{1\cdot 2\cdots\nu}=m^n. \qquad (8)$$

The formula obtained holds, whatever may be the positive integral value of $m$. One concludes this that this formula presents an identity.

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By comparing the coefficients of $m^n$ in the formula (8), one finds

$$\sum_{k=1}^{\sigma_n}\omega_k^n=n!.$$

It follows that

$$\sigma_n\leq n!.$$

Let us introduce in our studies the finite difference of different orders by defining them by the formula

$$\Delta^{(\mu)}f(m)=\sum_{k=0}^{\mu}(-1)^{\mu-k}\frac{\mu!}{k!(\mu-k)!}f(m+\mu).$$

The formula (8) gives

$$\sum_{\nu=\mu}^{n}\sum_{k=1}^{\sigma_\nu}\sum_{h=1}^{\omega_k^{(\nu)}}\frac{(m+\mu-a_{hk}^{(\nu)})\cdots(m+\nu-1-a_{hk}^{(\nu)})}{1\cdot 2\cdots(\nu-\mu)}=\Delta^{(\mu)}(m^n).$$

$$(\mu=0,1,2,\ldots,n)$$

By making $m=1$ in this formula and by noticing that

$$\frac{(\mu+1-a_{hk}^{(\nu)})\cdots(\nu-a_{hk}^{(\nu)})}{1\cdot 2\cdots(\nu-\mu)}\geq 0$$

since, because of (5) and (6)

$$a_{hk}^{(\nu)}\leq\nu,$$

one finds

$$\sum_{k=1}^{\sigma_\mu}\omega_k^{(\mu)}\leq\Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu=0,1,2,\ldots,n) \qquad (9)$$

It follows that

$$\sigma_\mu \leq \Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu = 0, 1, 2, \ldots, n)$$

We have seen in Number 60 that

$$S_\nu = (n+1-\nu)\sigma_{n-\nu}, \quad (\nu = 0, 1, 2, \ldots, n) \tag{10}$$

therefore

$$S_\nu \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n)$$

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Let us examine the conditions which have to be fulfilled for the symbols $S_\nu$ $(\nu = 0, 1, 2, \ldots, n)$ to be expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n) \tag{11}$$

By virtue of inequalities (9), it is necessary that

$$\omega_k^{(\mu)} = 1. \quad (k = 1, 2, \ldots, \sigma_\mu; \mu = 0, 1, 2, \ldots, n) \tag{12}$$

These are the conditions necessary and sufficient for the formula (11) to hold. In effect, in the case $\omega_k^{(\nu)} = 1$, the formula (7) becomes

$$m_k^{(\nu)} = \frac{(m-\nu)(m+1-\nu)\cdots(m-1)}{1 \cdot 2 \cdots \nu},$$

and the equality (8) takes the form

$$\sum_{\nu=0}^{n} \sigma_\nu \frac{(m-\nu)\cdots(m-1)}{1 \cdot 2 \cdots \nu} = m^n.$$

It follows that

$$\sum_{\nu=\mu}^{n} \sigma_\nu \frac{(m+\mu-\nu)\cdots(m-1)}{1 \cdot 2 \cdots (\nu-\mu)} = \Delta^{(\mu)}(m^n),$$

and by making $m = 1$, one obtains

$$\sigma_\mu = \Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu = 0, 1, 2, \ldots, n)$$

It results in, because of (10), the formula (11).

Let us notice that the conditions (12) come down to a single condition

$$\sigma_n = n!.$$

We will see that there exists primitive parallelohedra which satisfy this condition.

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*Theorem. The faces in 1, 2, 3 and 4 dimensions of simplexes of the set* $(L)$ *enjoy the property that*

$$\omega_k^{(}\nu) = 1. \quad (k = 1, 2, \ldots, \sigma_\nu; \nu = 1, 2, 3, 4)$$

The demonstration of the theorem introduced does not present difficulties.

*Corollary. The number of faces in different dimensions of primitive parallelohedra in the space of 2, 3 and 4 dimensions is expressed by the formula (11).*

1. By making in the formula (11) $n = 2$, one obtains

$$S_0 = 6 \text{ and } S_1 = 6.$$

2. By making in the formula (11) $n = 3$, one obtains

$$S_0 = 24, \;\; S_1 = 36, \;\; S_2 = 14.$$

3. By making in the formula (11) $n = 4$, one obtains

$$S_0 = 120, \;\; S_1 = 240, \;\; S_2 = 150, \;\; S_3 = 30.$$

By studying the primitive parallelohedra in the space of 5 dimensions, I have come across parallelohedra the number of faces of which is not expressed by the formula (11).

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We have seen that, in the case $\omega_k^{(\nu)} = 1$, one has

$$a_{kh}^{(\nu)} = \nu.$$

It is easy to demonstrate that, in the case $\omega_k^{(\nu)} > 1$, one will have this equality for a single set (3) which is composed of integers; for all the sets which remain, one will have the inequalities

$$2 \leq a_{hk}^{(\nu)} \leq \nu - 2. \;\; (\nu \geq 5) \tag{13}$$

Let us make in the formula (8) $m = 0$. By noticing that

$$\frac{(-a_{hk}^{(\nu)})(1 - a_{hk}^{(\nu)}) \cdots (\nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu} = 0$$

so long as $a_{hk}^{(\nu)} \neq \nu$, one finds

$$\sum_{\nu=0}^{n} (-1)^\nu \sigma_\nu = 0.$$

By making in the formula (8) $m = -1$, one obtains, because of (13),

$$\sum_{\nu=0}^{n}(-1)^{\nu}(\nu+1)\sigma_{\nu} = (-1)^{n}.$$

By substituting in this formula the expression of $\sigma_{\nu}$ derived from the formula (10), one will have

$$\sum_{\nu=0}^{n}(-1)^{\nu}S_{\nu} = 1. \tag{14}$$

Let us notice that the equality obtained expresses a property of faces in different dimensions of primitive parallelohedra which is common to all the convex polyhedra of the space in $n$ dimensions. † By making in the formula (14) $n = 3$, one will have

$$S_0 - S_1 + S_2 - S_3 = 1,$$

and as $S_3 = 1$, this becomes

$$S_0 + S_2 = 2 + S_1.$$

This is the well known formula of Euler. ‡

*Regulators and characteristics of edges of primitive parallelohedra.*

69

Let us examine the set $(R)$ of primitive parallelohedra belonging to a type of parallelohedra characterised by a set $(L)$ of simplexes.

Let $(\alpha_i)$ be a vertice of parallelohedra of the set $(R)$ determined by the equations

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = A. \quad (k = 1, 2, \ldots, n) \tag{1}$$

The system $L$ correlative to the vertice $(\alpha_i)$ is characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}). \tag{2}$$

Let us indicate by $(\alpha_{ik}$ $(k = 0, 1, 2, \ldots, n))$ the vertices adjacent to the vertice $(\alpha_i)$ (Number 18). The simplex $L_k$ $(k = 0, 1, 2, \ldots, n)$

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† See: *Poincaré*, Sur la généralisation d'un théorème d'Euler relatif aux polyèdres. [On the generalisation of the theorem of Euler relative to the polyhedra] (Comptes Rendus des Séances de l'Académie de Paris, V. 117, p. 144)

‡ Euler, Elementa doctrinae Solidorum. (Novi Comment. Petrop. 1758.)

correlative to the vertice $(\alpha_{ik})$ will be characterised by the systems which one obtains from system (2) by replacing the vertice $(l_{ik})$ of the simplex $L$ by a corresponding vertice $(l'_{ik})$ of the simplex $L_k$. The two simplexes $L$ and $L_k$ are contiguous by a face in $n-1$ dimensions $P_k$ $(k = 0, 1, 2, \dots, n)$ which is characterised by the systems

$$(l_{ih}). \quad (k = 0, 1, 2, \dots, n; h \neq k)$$

The face $P_k$ of simplexes $L$ and $L_k$ is correlative to an edge $[\alpha_i, \alpha_{ik}]$ of parallelohedra of the set $(R)$.

Let us indicate by

$$\sum p_{ik} x_i = \delta_k, \quad (k = 0, 1, 2, \dots, n)$$

the equation of the face $P_k$. As one has

$$\sum p_{ik} l_{ik} = \delta_k, \quad (h = 0, 1, 2, \dots, n; h \neq k)$$

it becomes

$$\sum p_{ik}(l_{ik} - l_{ir}) = 0. \; (h = 0, 1, 2, \dots, n; r = 0, 1, 2, \dots, n; h \neq k; r \neq k) \tag{3}$$

The equalities obtained define the number $p_{1k}, p_{2k}, \dots, p_{nk}$ to a common factor close by. By supposing that $p_{1k}, p_{2k}, \dots, p_{nk}$ be integer not having common divisor, one will determine by the equality (3) two systems $(p_{ik})$ and $(-p_{ik})$. One will call characteristic of the edge $[\alpha, \alpha_{ik}]$ or of the correlative face $P_k$ one of the two systems $\pm(p_{ik})$ likewise.

By noticing that

$$\sum p_{ik} l_{ik} \neq \delta_k,$$

one will attach, for more precision, a supplementary condition

$$\sum p_{ik} l_{ik} > \delta_k.$$

*Definition. One will call characteristic of the face $P_k$ with regard to the simplex L the system $(p_{ik})$ which is well defined by the conditions*

$$\sum p_{ik} l_{ik} > \delta_k, \; \sum p_{ik} l_{ih} = \delta_k. \; (h = 0, 1, 2, \dots, n, h \neq k) \tag{4}$$

Let us notice that the characteristic of the face $P_k$ with regard to the simplex $L_k$ will be the system $(-p_{ik})$. In effect, one will have

$$\sum p_{ik} l'_{ik} \neq \delta_k.$$

Let us suppose that

$$\sum p_{ik} l'_{ik} > \delta_k.$$

In this case the two simplexes $L$ and $L_k$ would be situated on the same side of the face $P_k$, and one could find a point interior to the simplex $L$ which would be interior to the simplex $L_k$ too, this is contrary to Theorem I demonstrated in Number (6). It is therefore necessary that

$$\sum p_{ik} l'_{ik} < \delta_k,$$

and the system $(-p_{ik})$ presents the characteristic of the face $P_k$ with regard to the simplex $L_k$.

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One will determine the vertice $(\alpha_{ik})$ $(k = 0, 1, 2, \dots, n)$ correlative to the simplex $L_k$ by the equations

$$\sum\sum a_{ij} l_{ih} l_{jh} + 2\sum \alpha_{ik} l_{ih} = A_k \quad (h = 0, 1, 2, \dots, n; h \neq k) \qquad (5)$$

and

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_{ik} l'_{ik} = A_k. \qquad (6)$$

From the equalities (1) and (5), one derive

$$2\sum(\alpha_{ik} - \alpha_i) l_{ih} = A_k - A. \quad (h = 0, 1, 2, \dots, n; h \neq k) \qquad (7)$$

As a result because of (3), one will have

$$\alpha_{ik} - \alpha_i = p_{ik}\rho_k. \quad (i = 1, 2, \dots, n; k = 0, 1, 2, \dots, n) \qquad (8)$$

On the ground of the supposition made, the vertices $(\alpha_i)$ and $(\alpha_{ik})$ $(k = 0, 1, 2, \dots, n)$ of primitive parallelohedra of the set $(R)$ are simple.

It follows that,

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_{ik} l'_{ik} > A$$

and

$$\sum\sum a_{ij} l_{ik} l_{i[\text{sic}]k} + 2\sum \alpha_{ik} l_{ik} > A_k.$$

By virtue of (1) and (6), one obtains

$$2\sum(\alpha_{ik} - \alpha_i) l_{ik} > A_k - A \text{ and } 2\sum(\alpha_{ik} - \alpha_i) l'_{ik} < A_k - A$$

and, because of (8), it becomes

$$2\rho_k \sum p_{ik} l_{ik} > A_k - A \text{ and } 2\rho_k \sum p_{ik} l'_{ik} < A_k - A. \qquad (9)$$

As by virtue of (7) and (8), one has

$$2\rho_k \sum p_{ik} l_{ih} = A_k - A, \quad (h = 0, 1, 2, \dots, n; h \neq k) \tag{10}$$

the inequalities (9) can be written

$$2\rho_k \sum p_{ik}(l_{ik} - l_{ih}) > 0, \; 2\rho_k \sum p_{ik}(l'_{ik} - l_{ih}) < 0. \tag{11}$$

$$(h = 0, 1, 2, \dots, n; h \neq k)$$

By noticing that because of (4)

$$\sum p_{ik}(l_i - l_{ih}) > 0, \quad (h = 0, 1, 2, \dots, n; h \neq k) \tag{12}$$

one finds

$$\rho_k > 0, \quad (k = 0, 1, 2, \dots, n) \tag{13}$$

and the second inequality (11) gives

$$\sum p_{ik}(l'_{ik} - l_{ih}) < 0, \quad (h = 0, 1, 2, \dots, n; h \neq k) \tag{14}$$

or differently, because of (4),

$$\sum p_{ik} l'_{ik} < \delta_k, \quad (k = 0, 1, 2, \dots, n) \tag{15}$$

that which we have demonstrated by another method.

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By substituting in (6) the expression of $\alpha_{ik}$ derived from the equality (8), one obtains

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_i l'_{ik} + 2\rho_k \sum p_{ik} l'_{ik} = A_k.$$

One will present this equality, because of (1), under the form

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_i l'_{0k} - \sum\sum a_{ij} l_{ik} l_{jk} - 2\sum \alpha_i l_{ik} = A_k - A - 2\rho_k \sum p_{ik} l'_{ik},$$

and lastly, by virtue of (10),

$$\begin{aligned} 2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = & \sum\sum a_{ij} l'_{ik} l_{jk} + 2\sum \alpha_i l'_{ik} \\ & - \sum\sum \alpha_{ij} l_{ik} l_{jk} - 2\sum \alpha_i l_{ik} \end{aligned} \tag{16}$$

where $h = 0, 1, 2, \dots, n, h \neq k, k = 0, 1, 2, \dots, n$.

*Definition II. One will call regulator of the edge $[\alpha_i, \alpha_{ik}]$ or of the correlative face $P_k$ the positive parameter $\rho_k$ defined by the formulae (8) and (16).*

Let us notice that on the ground of equalities (3) and (8) the congruent edges and the congruent correlative faces possess the same regulator and the same characteristic.

One can determine the regulator $\rho_k$ by other formulae.

Let us write

$$l'_{ik} = \sum_{r=0}^{n} \vartheta_r^{(k)} l_{ir} \text{ where } \sum_{r=0}^{n} \vartheta_r^{(k)} = 1. \ (k = 1, 2, \dots, n) \tag{17}$$

On the grounds of equations (1) and (17), one obtains

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_i l'_{ik} - \sum\sum a_{ij} l_{ik} l_{jk} - 2\sum \alpha_i l_{ik} = \sum\sum a_{ij} l'_{ik} l'_{jk} - \sum_{r=0}^{n} \vartheta_r^{(k)} \sum\sum a_{ij} l_{ir} l_{jr}.$$

By substituting in the formula (16), one finds

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = \sum\sum a_{ij} l'_{ik} l'_{jk} - \sum_{r=0}^{n} \vartheta_r^{(k)} \sum\sum a_{ij} l_{ir} l_{jr} \tag{18}$$

where $h = 0, 1, 2, \dots, n; h \neq k; k = 0, 1, 2, \dots, n$.

The formula obtained makes visible an important property of the regulator $\rho_k$: *the regulator $\rho_k$ is expressed by a linear function of coefficients of the quadratic form $\sum\sum a_{ij} x_i x_j$.* By writing

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij}, \tag{19}$$

one will have the rational coefficients $p_{ij}^{(k)} = p_{ji}^{(k)}$, $i = 1, 2, \dots, n, j = 1, 2, \dots, n$.

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By virtue of the formula (19), the regulator $\rho_k$ will be perfectly determined if one knows the corresponding coefficients $p_{ij}^{(k)}$ $(i = 1, 2, \dots, n; j = 1, 2, \dots, n)$.

As the coefficients $a_{ij}$ $(i = 1, 2, \dots, n; j = 1, 2, \dots, n)$ of the quadratic form $\sum\sum a_{ij} x_i x_j$ do not play any role in the determination of coefficients $p_{ij}^{(k)}$ $(i = 1, 2, \dots, n; j = 1, 2, \dots, n)$ which depend only on the simplexes $L$ and $L_k$, one can replace in the previous formula the coefficients $a_{ij}$ by the coefficients $x_i x_j$ $(i = 1, 2, \dots, n; j = 1, 2, \dots, n)$.

By introducing the linear functions, as we have done in Number 58,

$$u_r = \sum l_{ir} x_i, \ v_r = \sum l'_{ir} x_i, \quad (r = 0, 1, 2, \dots, n)$$

let us indicate by

$$u_r^{(k)} \text{ and } v_r^{(k)} \quad (r = 0, 1, 2, \ldots, n)$$

the values of these functions which correspond to the values of variables $x_1, x_2, \ldots, x_n$

$$x_i = p_{ik}. \quad (k = 0, 1, 2, \ldots, n)$$

By virtue of (4), one will have

$$u_h^{(k)} = \delta_k. \quad (h = 0, 1, 2, \ldots, n; h \neq k, u_k^{(k)} > \delta_k)$$

By virtue of (15), one will have

$$v_k^{(k)} < \delta_k.$$

Let us notice that the numbers $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ defined by the equality (7) will be determined by the equalities

$$v_k = \sum_{r=0}^{n} \vartheta_r^{(k)} u_r \text{ where } \sum \vartheta_r^{(k)} = 1.$$

By replacing in the formula (18) the coefficients $a_{ij}$ by the coefficients $x_i x_j$, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$, one obtaines

$$2\rho_k(\delta_k - v_k^{(k)}) = (v_k)^2 - \sum_{r=0}^{n} \vartheta_r^{(k)} (u_r)^2. \quad (k = 0, 1, 2, \ldots, n) \qquad 20$$

To return the formula obtained to the formula (19), it suffices to replace in the equality

$$\rho_k = \sum \sum p_{ij}^{(k)} x_i x_j$$

the coefficients $x_i x_j$ by the coefficients $a_{ij}$, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

*Fundamental transformation of the form*

$$\sum \sum a_{ij} x_i x_j + 2 \sum \alpha_i x_i - \sum \sum a_{ij} l_i l_j - 2 \sum \alpha_i l_i$$

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By keeping the previous notations, let us indicate

$$F_{(L)}(x_1, x_2, \ldots, x_n) = \sum \sum a_{ij} x_i x_j + 2 \sum \alpha_i x_i - A \qquad (1)$$

where one has admitted

$$A = \sum \sum a_{ij} l_{ik} l_{jk} + 2 \sum \alpha_i l_{ik}. \quad (k = 0, 1, 2, \ldots, n) \qquad (2)$$

By introducing the variables $\xi_0, \xi_1, \ldots, \xi_n$ after the conditions

$$x_i = \sum_{r=0}^{n} \xi_r l_{ir} \text{ where } \sum_{r=0}^{n} \xi_r = 1, \tag{3}$$

one will present the function $F_{(L)}\ (x_1, x_2, \ldots, x_n)$ under the following form:

$$F_{(L)}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij} x_i x_j - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij} l_{ir} l_{jr}. \tag{4}$$

One concludes that the function $F_{(L)}\ (x_1, x_2, \ldots, x_n)$ is linear with regard to the coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ of an arbitrary quadratic form $\sum\sum a_{ij} x_i x_j$.

By making in the formula (1) $x_i = l_{ik}$, one obtains, because of (2),

$$F_{(L)}(l_{1k}, l_{2k}, \ldots, l_{nk}) = 0. \quad (k = 0, 1, 2, \ldots, n)$$

The equalities obtained hold, whatever may be the values of $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

Let us indicate by $L^0$ a simplex congruent to the simplex $L$ and characterised by the vertices

$$(l_{i0} + l_i), (l_{i1} + l_i), \ldots, (l_{in} + l_i),$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers.

By noticing that because of (3)

$$x_i + l_i = \sum_{r=0}^{n} \xi_r (l_{ir} + l_i) \text{ where } \sum_{r=0}^{n} \xi_r = 1,$$

one will have an equality

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) =$$

$$\sum\sum a_{ij}(x_i + l_i)(x_j + l_j) - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij}(l_{ir} + l_i)(l_{jr} + l_j)$$

and after the reductions, it becomes

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = \sum\sum a_{ij} x_i x_j - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij} l_{ir} l_{jr},$$

therefore, because of (4) one will have

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = F_{(L)}(x_1, x_2, \ldots, x_n). \tag{5}$$

By virtue of the formula (18) of Number 71, one will determine the regulator $\rho_k$ in the formula

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = F_{(L)}(l'_{1k}, l'_{2k}, \ldots, l'_{nk}). \ (k = 0, 1, 2, \ldots, n) \quad (7)$$

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Let us indicate

$$F_{L_k}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij}x_ix_j + 2\sum \alpha_{ik}x_i - A_k, \ (k = 0, 1, 2, \ldots, n)$$

$L_k$ being a simplex contiguous to the simplex $L$ by the face $P_k$ ($k = 0, 1, 2, \ldots, n$).

By substituting in this equality the expression of $\alpha_{ik}$ defined from the formula (8) of Number 70, one obtains

$$F_{L_k}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i + 2\rho_k \sum p_{ik}x_i - A_k$$

and, because of (1), one will have

$$F_{L_k}(x_1, x_2, \ldots, x_n) = F_{(L)}(x_1, x_2, \ldots, x_n) + 2\rho_k \sum p_{ik}x_i + A - A_k.$$

By substituting in this equality the expression of $A - A_k$ given by the formula (10) of Number 70, one finds

$$F_{L_k}(x_1, x_2, \ldots, x_n) = F_{(L)}(x_1, x_2, \ldots, x_n) + 2\rho_r \sum p_{ik}(x_i - l_{ih}). \ (h \neq k)$$

This formula can be written

$$F_{(L)}(x_1, x_2, \ldots, x_n) = F_{L_k}(x_1, x_2, \ldots, x_n) + 2\rho_k \sum p_{ik}(l_{ih} - x_i). \quad (*)$$
$$(h \neq k, k = 0, 1, 2, \ldots, n)$$

The formula (*) obtained is capable of numerous and important applications.

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Let us suppose that $x_1, x_2, \ldots, x_n$ be arbitrary integers and that the point $(x_i)$ is not found among the vertices

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \quad (7)$$

of the simplex $L$. By admitting

$$x_i = \sum_{r=0}^{n} \xi_r l_{ir} \ \text{ where } \ \sum_{r=0}^{n} \xi_r = 1, \quad (8)$$

one will have by virtue of Theorem II of Number 62 among the numbers $\xi_0, \xi_1, \dots, \xi_n$ at least one negative number. Let us suppose to fix the ideas that

$$\xi_k < 0. \tag{9}$$

By noticing that because of (3) and of the formula (4) in Number 69 one has

$$\sum p_{ik}(l_{ih} - x_i) = \xi_k \sum p_{ik}(l_{ih} - l_{ik})$$

and that

$$\sum p_{ik}(l_{ih} - l_{ik}) < 0, \quad (h \neq k)$$

one obtains, because of (9),

$$\sum p_{ik}(l_{ih} - x_i) > 0.$$

One concludes that the coefficient of $2\rho_k$ in the formula (*) is an integer and positive in the case considered.

In the same manner, one will examine the function $F_{L_k}(x_1, x_2, \dots, x_n)$ and so on.

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Let us suppose that one have examined the simplex,

$$L, L', L'', \dots, L^{(m)} \tag{10}$$

successively contiguous by the faces in $n-1$ dimensions the regulator of which present the function

$$\rho_1, \rho_2, \dots, \rho_m.$$

Let us suppose that by applying the formula (*) to the simplexes (10) one have obtained the equalities

$$\begin{aligned}
F_L(x_1, x_2, \dots, x_n) &= F_{L'}(x_1, x_2, \dots, x_n) + 2h_1\rho_1 \text{ where } h_1 > 0,\\
F_{L'}(x_1, x_2, \dots, x_n) &= F_{L''}(x_1, x_2, \dots, x_n) + 2h_2\rho_2 \text{ where } h_2 > 0,\\
&\dots\\
F_{L^{(m-1)}}(x_1, x_2, \dots, x_n) &= F_{L^{(m)}}(x_1, x_2, \dots, x_n) + 2h_m\rho_m \text{ where } h_m > 0.
\end{aligned}$$

It follows that

$$F_L(x_1, x_2, \dots, x_n) = 2\sum_{k=1}^{m} h_k\rho_k + F_{L^{(m)}}(x_1, x_2, \dots, x_n). \tag{11}$$

The procedure shown can not be prolonged indefinitely and one will always arrive at a simplex $L^{(m)}$ among the vertices of which is found the point $(x_i)$.

To demonstrate this, let us notice that the coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ of the quadratic form $\sum\sum a_{ij}x_i x_j$ in the formulae obtained are arbitrary.

Let us suppose that one have chosen the positive quadratic form $\sum\sum a_{ij}x_i x_j$ which defines a set $(R)$ of primitive parallelohedra belonging to the type characterised by the set $(L)$ of simplexes.

We have seen in Number 70 that one will have the inequalities

$$\rho_k > 0. \quad (k = 1, 2, \ldots, m)$$

By virtue of the definition of the function $F_L(x_1, x_2, \ldots, x_n)$, one will have an inequality

$$F_L(x_1, x_2, \ldots, x_n) > 0,$$

whatever the integral values of $x_1, x_2, \ldots, x_n$ may be, abstraction made from vertices (7) of the simplex $L$. It results in

$$F_{(L^{(m)})}(x_1, x_2, \ldots, x_n) \geq 0$$

and the formula (11), in the case considered, gives

$$F_L(x_1, x_2, \ldots, x_n) \geq 2 \sum_{k=1}^{m} h_k \rho_k.$$

As the coefficients $h_k$ $(k = 1, 2, \ldots, m)$ are of positive integers and the regulators $\rho_k$ $(k = 1, 2, \ldots, m)$ belong to a series of regulators corresponding to the incongruent faces of simplexes of the set $(L)$, one concludes that the number $m$ can not be increased indefinitely. As a result the series (10) will be terminated by a simplex $L^{(m)}$ among the vertices of which is found the point $(x_i)$.

It follows that one will have indentically

$$F_{(L^{(m)})}(x_1, x_2, \ldots, x_n) = 0,$$

and the formula (11) becomes

$$F_L(x_1, x_2, \ldots, x_n) = 2 \sum_{[k]=1}^{m} h_k \rho_k \text{ where } (k = 1, 2, \ldots, m) \qquad (12)$$

Let us notice that the formula obtained presents an identity which holds, whatever the values of coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ may be, provided that the regulators $\rho_k$ $(k = 1, 2, \ldots, m)$ are expressed by the formula (6).

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*Fundamental theorem. Let us suppose that the regulators* $\rho_k$ $(k = 1, 2, \ldots, \sigma)$ *corresponding to the various incongruent faces in* $n - 1$

*dimensions of simplexes belonging to the set* $(L)$ *be determined by the equations*

$$\rho_k = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}^{(k)} a_{ij}. \quad (k = 1, 2, \ldots, \sigma)$$

*For a quadratic form* $\sum \sum a_{ij} x_i x_j$ *to define a set* $(R)$ *primitive parallelohedra belonging to the type characterised by the set* $(L)$ *of simplexes, it is necessary and sufficient that the inequalities*

$$\rho_k = \sum \sum p_{ij}^{(k)} a_{ij} > 0, \quad (k = 1, 2, \ldots, \sigma)$$

*hold.*

We have seen in Number 70 that the inequalities

$$\rho_k > 0, \ \ (k = 1, 2, \ldots, \sigma) \qquad 13$$

present the necessary conditions. Let us suppose the coefficients of a quadratic form $\sum \sum a_{ij} x_i x_j$ verify the inequalities (13). By virtue of the formula (12), one will have the inequality

$$F_{(L)}(x_1, x_2, \ldots, x_n) > 0$$

so long as the point $(x_i)$ the element s of which are integers is not found among the vertices of the simplex $L$. By virtue of the definition established, the simplex $L$ is in this case correlative to a simplex vertice $(\alpha_i)$ of parallelohedra corresponding to the quadratic form examined $\sum \sum a_{ij} x_i x_j$.

The simplex $L$ is chosen arbitrary in the set $(L)$ of simplexes, therefore all the simplexes of the set $(L)$ are correlative to the simple vertices of parallelohedra corresponding to the quadratic form $\sum \sum a_{ij} x_i x_j$.

I argue that these parallelohedra do not possess other vertices, it is that which one will verify without trouble.

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Let us notice that any quadratic form $\sum \sum a_{ij} x_i x_j$ verifying the inequalities (13) is positive. To demonstrate this, let us examine a simplex $L$ among the vertices in which is found the point (0). One will have in this case

$$F_{(L)}(x_1, x_2, \ldots, x_n) = \sum \sum a_{ij} x_i x_j + 2 \sum \alpha_i x_i,$$

and consequently

$$F_{(L)}(x_1, x_2, \ldots, x_n) + F_{(L)}(-x_1, -x_2, \ldots, -x_n) = 2 \sum \sum a_{ij} x_i x_j.$$

The two points $(x_i)$ and $(-x_i)$ can not be the vertices of the simplex $L$, the point $(0)$ being excluded. This results in

$$F_{(L)}(x_1, x_2, \ldots, x_n) + F_{(L)}(-x_1, -x_2, \ldots, -x_n) > 0,$$

therefore

$$\sum\sum a_{ij}x_ix_j > 0,$$

whatever may be the integral values of $x_1, x_2, \ldots, x_n$, the system $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ being excluded.

*Definition of quadratic forms with the help of regulators and corresponding characteristics.*

79 Let us take any one quadratic form $\sum\sum a_{ij}x_ix_j$ in arbitrary coefficients. Let us choose $n$ numbers $x_1, x_2, \ldots, x_n$ which are subject to the only condition: *the equality*

$$h_1x_2 + h_2x_2 + \ldots + h_nx_n = 0$$

*is impossible so long as the numbers* $h_1, h_2, \ldots, h_n$ *are integers.*

Let us examine a vector $g$ made up of points

$$\frac{l_1}{m} + ux_i \text{ where } 0 \leq u \leq 1,$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers and $m$ being any one positive integer.

The vector $g$ will traverse a certain number of simplexes belonging to the set $(L)$. Let us indicate by

$$L_0, L_1, \ldots, L_s \tag{1}$$

the simplexes of the set $(L)$ which contain the various parts of the set $g$. On the ground of the supposition made, the simplexes (1) are well defined by the vector $g$ and are successively contiguous by the faces in $n-1$ dimensions. In effect, two adjacent simplexes $L_k$ and $L_{k+1}$ of the series (1) possess a common point $(\xi_{ik})$ belonging to the vector $g$, therefore the simplexes $L_k$ and $L_{k+1}$ are contiguous by a face in any number of dimensions. Let us suppose that this face be characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{i\nu}). \tag{2}$$

As

$$\xi_{ik} = \frac{l_i}{m} + u_kx_i \text{ where } 0 < u_k < 1,$$

one will have

$$\frac{l_i}{m} + u_k x_i = \sum_{r=0}^{\nu} \vartheta_r l_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r > 0. \ (r = 0, 1, 2, \dots, \nu) \tag{3}$$

By supposing that $\nu < n-1$, one wil determine with the help of these equalities a system $(h_i)$ of integers verifying the equation

$$h_1 x_1 + h_2 x_2 + \dots + h_n x_n = 0,$$

which is contrary to the hypothesis, therefore it is necessary that $\nu = n-1$ and that the point $(\xi_{ik})$ be interior to a face in $n-1$ dimensions which is common to the simplexes $L_k$ and $L_{k+1}$.

Let us suppose that $\nu = n-1$. By indicating with $(p_{ik})$ the characteristic of the face $(P_k)$ characterised by the systems (2) with regard to the simplex $L_k$, one will have, by virtue of the formula (4) of Number 69,

$$\sum p_{ik} l_{ir} = \delta_k, \quad (r = 0, 1, 2, \dots, n-1)$$

and the equalities (3) give

$$\sum p_i (\frac{l_i}{m} + u_k x_i) = \delta_k$$

and consequently

$$u_k \sum p_{ik} x_i = \delta - \sum p_{ik} \frac{l_i}{m}.$$

As $\sum p_{ik} x_i \neq 0$, on the ground of the supposition made, the equality obtained defines a point $(\xi_{ik})$ of the vector $g$ which is interior to the face $P_k$. This results in that the vector $g$ does not possess other points common to the face $P_k$. By attributing to the parameter $u$ a negative variation $\delta u$ sufficiently small, one will define a point $\frac{l_i}{m} + (u_k + \delta u) x_i$ of the vector $g$ which is interior to the simplex $L_k$. By attributing to the parameter $u$ a variable $\delta u > 0$, one obtains a point $\frac{l_i}{m} + (u_k + \delta u) x_i$ which is interior to the simplex $L_{k+1}$.

As in these cases one has

$$\sum p_{ik} (\frac{l_i}{m} + (u_k + \delta u) x_i) > \delta_k, \quad (\delta u < 0)$$

and

$$\sum p_{ik} (\frac{l_i}{m} + (u_k + \delta u) x_i) < \delta_k, \quad (\delta u > 0)$$

it becomes

$$\sum p_{ik}x_i < 0. \tag{4}$$

By indicating with $\rho_k$ the regulator of the face $P_k$ with regard to the chosen quadratic form $\sum a_{ij}x_ik_j$ $(k = 0, 1, 2, \ldots, s-1)$, let us apply the formula (*) of Number 74 to the simplexes (1). One will have the equalities

$$\begin{aligned} F_{(L_0)}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) &= F_{L_1}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) \\ &\quad + 2\rho_0 \sum p_{i0}(l_{ih_0} - \frac{l_i}{m} - x_i), \\ &\ldots \\ F_{L_{s-1}}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) &= F_{L_s}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) \\ &\quad + 2\rho_{s-1} \sum p_{i,s-1}(l_{ih_{s-1}} - \frac{l_i}{m} - x_i). \end{aligned}$$

It follows that

$$\begin{aligned} F_{(L_0)}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) &= 2\sum_{k=0}^{s-1} \rho_k \sum i = 1^n p_{ik}(l_{ih_k} - \frac{l_i}{m} - x_i) \\ &\quad + F_{L_s}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n). \end{aligned} \tag{5}$$

Until now, the integers $l_1, l_2, \ldots, l_n$ had been arbitrary. Let us suppose that the integers $l_1, l_2, \ldots, l_n$ satisfy the conditions

$$0 \leq l_i < m. \ (i = 1, 2, \ldots, n) \tag{6}$$

Let us indicate by $K$ the set of incongruent points $(\frac{l_i}{m})$ verifying these inequalities.

The number of points belonging to the set $K$ is equal to $m$.

Let us apply the formula (5) to all the points of the set $K$ and make the sum of equalities obtained. One will have a formula

$$\begin{aligned} &F_{(L_0)}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) = \\ &2\sum \rho_k \sum p_{ik}(l_{ih_k} - \frac{l_i}{m} - x_i) + \sum F_{(L_s)}(\frac{l_1}{m} + x_i, \ldots, \frac{l_n}{m} + x_n). \end{aligned} \tag{7}$$

All the sums which are formed in this formula can be determined with a certain approximation.

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Let us suppose that the simplex $L_0$ be characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}).$$

On the ground of the supposition made, the point $\left(\frac{l_i}{m}\right)$ belongs to the simplex $L_0$. As there exist only a finite number of simplexes of the set $L$ to which belong the points $\left(\frac{l_i}{m}\right)$ verifying the inequalities (6), one concludes that one can determine a positive parameter $\lambda$ in such a manner that the inequalities

$$|l_{ik}| \leq \lambda \ \ (i = 1, 2, \ldots, n; k = 0, 1, 2, \ldots, n) \tag{8}$$

holds.

In this case, the corresponding value of the function

$$F_{(L_0)}\left(\frac{l_1}{m} + x_1, \frac{l_2}{m} + x_2, \ldots, \frac{l_n}{m} + x_n\right)$$

can be presented under the form

$$F_{(L_0)}\left(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n\right) = \sum\sum a_{ij}x_ix_j + \epsilon_0 + \sum \epsilon_i x_i$$

where the coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ do not exceed in numerical value a fixed limit $\epsilon$ which does not depend on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ and on the choice of the set $(L)$ of simplexes.

Let us examine the function $F_{(L_s)}\left(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n\right)$. After the proposition made, the point $\left(\frac{l_i}{m} + x_i\right)$ belongs to the simplex $L_s$. Let us determine the integers $t_1, t_2, \ldots, t_n$ after the conditions

$$0 \leq \frac{l_i}{m} + x_i + t_i < 1, \ \ (i = 1, 2, \ldots, m) \tag{9}$$

and indicae by $L'_s$ the simplex congruent to the simplex $L_s$ which obtains by a translation of the simplex $L_s$ the length of the vector $[t_i]$.

By virtue of the formula (5) of Number 73, one will have

$$F_{(L_s)}\left(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n\right) = F_{L'_s}\left(\frac{l_1}{m} + x_1 + t_1, \ldots, \frac{l_n}{m} + x_n + t_n\right). \tag{10}$$

By indicating with

$$(l'_{i0}), \ldots, (l'_{in})$$

the vertices of the simplex $L'_s$, one will have, by virtue of (9), the inequalities (8):

$$|l'_{ik}| \leq \lambda. \quad (i = 1, 2, \ldots, n; k = 0, 1, 2, \ldots, n)$$

It follows that the numerical value of the function

$$F_{L'_s}\left(\frac{l_1}{m}+x_1+t_1,\dots,\frac{l_n}{m}+x_n+t_n\right),$$

because of (8) and (9), does not exceed a fixed limit $\varphi$ which depends only on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ and on the choice of the set $(L)$ of simplexes. By virtue of (10), one can write

$$F_{(L_s)}\left(\frac{l_1}{m}+x_1,\dots,\frac{l_n}{m}+x_n\right)=\varphi_0 \text{ where } |\varphi_0|\le\varphi.$$

By substituting in the formula (7) the results obtained, one will present it under the following form

$$m_n\left(\sum\sum a_{ij}x_ix_j+\epsilon_0+\sum\epsilon_ix_i\right)=2\sum\rho_k\sum_{i=1}^{n}p_{ik}\left(l_{ih}-\frac{l_i}{m}-x_i\right) \tag{11}$$

In this formula, the coefficients $\epsilon_0,\epsilon_1,\dots,\epsilon_n$ do not exceed in numerical value a fixed limit which does not depend on numbers $x_1,x_2,\dots,x_n$.

81 Let us determine the coefficients of $2\rho_k$ in the formula obtained.

To that effect, let us choose any one face $P$ in $n-1$ dimensions of simplexes of the set $(L)$. Let us suppose that the face $P$ is characterised by the systems

$$(0),(l_{(i1)}),\dots,(l_{(i,n-1)}).$$

Let us indicate by $\rho$ the regulator and by $\pm(p_i)$ the characteristic of the face $P$ with regard to the quadratic form $\sum\sum a_{ij}x_ix_j$. One will suppose, on the ground of (4), that

$$\sum p_ix_i<0. \tag{12}$$

Let us suppose that the vector $g$ is made up of points

$$\frac{l_i}{m}+ux_i \text{ where } 0\le u\le 1$$

and corresponding to a point $\left(\frac{l_i}{m}\right)$ belonging to the set $K$ possesses a point which is interior to a face $P'$ congruent to the face $P$.

By supposing that the face $P'$ is characterised by the systems

$$(g_i),(l_{i1}+g_i),\dots,(l_{i,n-1},g_i),$$

one will have, on the ground of the supposition made,

$$\frac{l_i}{m} + ux_i = \sum_{k=0}^{n-1} \vartheta_k (g_i + l_{ik}) \text{ where } \sum_{k=0}^{n-1} \vartheta_k = 1 \text{ and } \vartheta_k > 0.$$

The corresponding value of the coefficient of $2\rho$ in the formula (11) is expressed by the sum

$$\sum \sum_{i=1}^{n} p_i (l_{ih} - \frac{l_i}{m} - x_i) \tag{14}$$

which extends to all the faces $P'$ congruent to the face $P$ verifying the equalities (13), provided that the points $\left(\frac{l_i}{m}\right)$ belong to the set $K$.

Let us indicate, to make short,

$$\sum_{i=1}^{n} p_i x_i = -\Delta \tag{15}$$

One will have, because of (12),

$$\Delta > 0.$$

As the system $(l_{ih})$ in the sum (14) indicate any one vertice of the face $P'$, one can write down

$$l_{ih} = g_i,$$

and the equalities (13) and (15) give

$$\sum_{i=1}^{n} p_i \left( g_i - \frac{l_i}{m} - x_i \right) = (1-u)\Delta.$$

Therefore, the study of the coefficient of $2\rho$ in the formula (11) comes down to the evaluation of the sum

$$\sum (1-u)\Delta \text{ where } 0 < u < 1. \tag{16}$$

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Let us designate, to summarise,

$$-mg_i + l_i = h_i. \tag{17}$$

The parameter $u$ verifying the equalities (13) is expressed by the formula

$$u = \frac{1}{m\Delta} \sum p_i h_i, \tag{18}$$

and as $0 < u < 1$, it becomes

$$0 < \sum p_i h_i < m\Delta.$$

By indicating with $\tau$ the integer verifying the inequalities

$$0 < \tau < m\Delta, \tag{19}$$

let us write

$$\sum p_i h_i = \tau. \tag{20}$$

By virtue of (18), the corresponding value of the parameter $u$ will be

$$u = \frac{\tau}{m\Delta}.$$

Let us substitute the expression found of the parameter $u$ in the equalities (13), it will become, because of (17),

$$\frac{\tau}{\Delta} x_i + h_i = m \sum_{k=1}^{n-1} \vartheta_k l_{ik} \text{ where } \sum_{k=1}^{n-1} \vartheta_k < 1 \text{ and } \vartheta_k > 0. \tag{21}$$

$$(k = 1, 2, \ldots, n-1)$$

This stated, let us notice that one can attribute to the number $\tau$ an arbitrary value verifying the inequalities (19). For similar values of $\tau$ to exist, it is necessary that

$$m\Delta > 1,$$

Let us suppose that the positive integer $m$ satisfies this condition. In this case, the finding of the sum (16) comes down to the solution of a sum

$$\sum (1-u)\Delta = \sum_{\tau>0}^{\tau<m\Delta} m_\tau (\Delta - \frac{\tau}{m}) \tag{22}$$

where $m_\tau$ indicates the number of systems $(h_i)$ of integers $h_1, h_2, \ldots, h_n$ verifying the equalities (20) and (21).

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It is easy to determine the number $m_\tau$.

Let us indicate by $(h_i^0)$ a system of integers verifying the equality

$$\sum p_i h_i^0 = 1. \tag{23}$$

As, on the ground of the supposition made, the integers $p_1, p_2, \ldots, p_n$ have no common divisor, the systems of integers verifying this equality always exist.

One will determine all the systems $(h_i)$ of integers verifying the equality (20) with the help of formulae

$$h_i = \tau h_i^0 + \sum_{k=1}^{n-1} \tau_k l_{ik} \tag{24}$$

where the rational numbers $\tau_1, \tau_2, \ldots, \tau_{n-1}$ belong to certain sets

$$\tau_k = \xi_{kr} + y_k, \quad (k = 1, 2, \ldots, n-1; r = 1, 2, \ldots, \omega) \tag{25}$$

$\omega$ being the greatest commondivisor of $n$ determinants of the order $(n-1)^2$ which one can form from $n-1$ systems

$$(l_{i1}), (l_{i2}), \ldots, (l_{i,n-1}).$$

By substituting the expressions of $h_1, h_2, \ldots, h_n$ derived from equalities (24) in the equalities (21), one obtains

$$\frac{\tau}{\Delta} x_i + \tau h_i^0 = \sum_{k-1}^{n-1} (m\vartheta_k - \tau_k) l_{ik}. \tag{26}$$

Let us notice that the numerical value of the determinant of $n$ systems

$$(x_i), (l_{i1}), \ldots, (l_{i,n-1})$$

is expressed by the formula

$$\pm \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ l_{11} & l_{21} & \cdots & l_{n1} \\ & \cdots & \cdots & \\ l_{1,n-1} & l_{2,n-1} & \cdots & l_{n,n-1} \end{vmatrix} = -\sum x_i p_i \omega = \omega \Delta.$$

Let us indicate by $\lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{nk}$, $(k = 1, 2, \ldots, n-1)$ the minor determinants which are defined by the equalities

$$\sum_{i=1}^{n} \lambda_{ik} l_{ik} = \omega \Delta, \quad (k = 1, 2, \ldots, n-1)$$

$$\sum \lambda_{ik} x_i = 0, \sum \lambda_{ik} l_{ir} = 0. \quad (r = 1, 2, \ldots, n-1; r \neq k)$$

The equalities (26) give

$$\tau \sum \lambda_{ik} h_i^0 = \omega \Delta (m\vartheta_k - \tau_k), \quad (k = 1, 2, \ldots, n-1)$$

and as a result

$$m\vartheta_k = \tau_k + \frac{\tau}{\omega \Delta} \sum \lambda_{ik} h_i^0. \quad (k = 1, 2, \ldots, n-1)$$

By virtue of (21) one obtains the inequalities

$$\tau_k + \frac{\tau}{\omega\Delta}\sum \lambda_{ik}h_i^0 > 0, \quad (k = 1, 2, \ldots, n-1)$$

$$\sum_{k=1}^{n-1}(\tau_k + \frac{\tau}{\omega\Delta}\sum \lambda_{ik}h_i^0) < m.$$

Considering the set (25), one finds

$$\begin{cases} y_k + \xi_{kr} + \dfrac{\tau}{\omega\Delta}\sum \lambda_{ik}h_i^0 > 0, \\ \displaystyle\sum_{k=1}^{n-1}\Big(y_k\xi_{kr} + \frac{\tau}{\omega\Delta}\sum \lambda_{ik}h_i^0\Big) < m. \end{cases} \quad (k = 1, 2, \ldots, n-1) \qquad (27)$$

Let us write

$$y_k\xi_{kr} + \frac{\tau}{\omega\Delta}\sum \lambda_{ik}h_i^0 = y'_k + \nu_k \text{ where } 0 < \nu_k \leq 1 \quad (k = 1, 2, \ldots, n-1)$$

and

$$\sum_{k=1}^{n-1}\nu_k = a_r^{(\tau)} + \nu \text{ where } 0 \leq \nu < 1, \qquad (28)$$

$y'_1, \ldots, y'_{n-1}$ and $a_r^{(\tau)}$ being integers.

The inequalities (27) will be replaced by the following ones:

$$\sum_{k=1}^{n-1} y'_k < m - a_r^{(\tau)} - \nu,\ y'_k > -\nu_k, \quad (k = 1, 2, \ldots, n-1)$$

or differently

$$\sum_{k=1}^{n-1} y'_k \leq m - a_r^{(\tau)} - 1 \text{ and } y'_k \geq 0. \quad (k = 1, 2, \ldots, n-1)$$

The number of systems $(y'_1, y'_2, \ldots, y'_{n-1})$ of integers verifying these inequalities is equal to

$$\frac{(m - a_r^{(\tau)})(m + 1 - a_r^{(\tau)}) \cdots (m + n - 2 - a_r^{(\tau)})}{1 \cdot 2 \cdots (n-1)}.$$

One concludes that the symbol $m_\tau$ which expresses the number of solutions of equations (20) and (21) in integers is equal to the sum

$$m_\tau = \sum_{\tau=1}^{\omega} \frac{(m - a_r^{(\tau)}) \cdots (m + n - 2 - a_r^{(\tau)})}{1 \cdot 2 \cdots (n-1)}.$$

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By substituting in the sum (26), one obtains

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)m_\tau=\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)\sum_{r=1}^{\omega}\frac{(m-a_r^{(\tau)})\cdots(m+n-2-a_r^{(\tau)})}{1\cdot 2\cdots(n-1)}. \tag{29}$$

Let us find a value approached by the sum obtained. By noticing that because of (28)

$$0\le a_r^{(\tau)}\le n-1,$$

one can write

$$\sum_{r=1}^{\omega}\frac{(m-a_r^{(\tau)})\cdots(m+n-2-a_r^{(\tau)})}{1\cdot 2\cdots(n-1)}=\frac{\omega}{(n-1)!}m^{n-1}+\delta_\tau m^{n-2}$$

where $|\delta_t au|$ does not exceed a fixed limit which doesnot depend on the number $m$.

By substituting in the sum (27), one finds

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)m_\tau=\frac{\omega}{(n-1)!}\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)+\delta\Delta^2m^{n-1}$$

where $|\delta|$ does not exceed a fixed limit which does not depend on $x_1, x_2, \ldots, x_n$ and on the number $m$.

By noticing that

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)=\Delta^2\frac{m}{2}-\frac{\Delta}{2}+\frac{\vartheta}{m}\quad\text{where } 0\le\vartheta<\frac{1}{8},$$

one can write

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)m_\tau=\frac{1}{2}\frac{\omega\Delta^2}{(n-1)!}m^n+m^{n-1}(\delta\Delta^2+\delta'\Delta+\delta'') \tag{30}$$

where $\delta, \delta', \delta''$ do not exceed in numerical value a fixed limit.

By substituting in the formula (11) the coefficient found of $2\rho$, one will have, because of (19),

$$m^n(\sum\sum a_{ij}x_ix_j+\epsilon_0+\sum\epsilon_ix_i)=\frac{m^n}{(m-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k(\sum p_ix_i)^2$$
$$+m^{n-1}\sum_{k=1}^{\sigma}2\rho_k[\delta_k(\sum p_{ik}x_i)^2+\delta'_k\sum p_{ik}x_i+\delta''_k]. \tag{31}$$

In the formula obtained the coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \delta_k, \delta'_k, \delta''_k$ $(k = 1, 2, \ldots, \sigma)$ do not exceed in numerical value a fixed limit which depend only on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ snf on the choice of the set $(L)$ of simplexes.

Let us replace in the formula obtained the numbers $x_1, x_2, \ldots, x_n$ by the numbers $mx_1, mx_2, \ldots, mx_n$. As these numbers satisfy the conditions imposed on the numbers $x_1, x_2, \ldots, x_n$, the formula (31) is applicable and one obtains

$$m^n(m^2\sum\sum a_{ij}x_ix_j + \epsilon_0 + m\sum \epsilon_i x_i) = \frac{m^{n-2}}{(n-1)!}\sum_{k=1}^{\sigma} \rho_k\omega_k(\sum p_{ik}x_i)^2$$

$$+ m^{n-1}\sum_{k=1}^{\sigma} 2\rho_k[\delta_k m^2(\sum p_{ik}x_i)^2 + \delta'_k m\sum p_{ik}x_i + \delta''_k].$$

By dividing the two parts of the formula obtained by $m^{n+2}$, let us make the positive integer $m$ increase indefinitely, it will become

$$\sum\sum a_{ij}x_ix_j = \frac{1}{(n-1)!}\sum_{k=1}^{\sigma} \rho_k\omega_k(p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{nk}x_n)^2. \tag{32}$$

The sum which is found in the second member of the formula obtained extends to all the incongruent faces in $n-1$ in $n-1$ dimensions of simplexes of the set $(L)$.

We have deduced the formula (34) by supposing that the numbers $x_1, x_2, \ldots, x_n$ form a irreducible basis. As the two parts of the formula (32) present two quadratic forms, one concludes that the formula (32) present an indentity. This results in that the formula (32) can be written

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a'_{ij} = \frac{1}{(n-1)!}\sum_{k=1}^{\sigma} \rho_k\omega_k \sum_{i=1}^{n}\sum_{j=1}^{n} a'_{ij}p_{ik}p_{jk} \tag{I}$$

where one has written

$$\rho_k = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}^{(k)} a_{ij}, \quad (k = 1, 2, \ldots, \sigma)$$

the two quadratic forms $\sum\sum a_{ij}x_ix_j$ and $\sum\sum a'_{ij}x_ix_j$ being arbitrary.

## Section V.

### Properties of the set $(\Delta)$ of quadratic forms

corresponding to the various types of primitive parallelohedra.

### *Definition of the domain of quadratic form corresponding to a type of primitive parallelohedra.*

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Let us suppose that a type of primitive parallelohedra is characterised by a set $(L)$ of simplexes.

Let us indicate by

$$\rho_k = \sum \sum p_{ij}^{(k)} a_{ij} \quad (k = 1, 2, \ldots, \sigma)$$

the regulators which correspond to the various incongruent faces in $n-1$ dimensions of simplexes of the set $(L)$.

*Definition. One will call domain of quadratic forms corresponding to the type of primitive parallelohedra characterised by the set* $(L)$ *of simplexes a domain* $\Delta$ *in quadratic forms verifying the inequalities*

$$\rho_k = \sum \sum p_{ij}^{(k)} a_{ij} \geq 0. \ (k = 1, 2, \ldots, \sigma) \qquad (1)$$

On the ground of the fundamental theorem of Number 77, for a quadratic form $f$ to define a set $(R)$ of primitive parallelohedra belonging to the type characterised by the set $(L)$ of simplexes, it is necessary and sufficient that the form $f$ is interior to the domain $\Delta$. This results in that the domain $\Delta$ is of $\frac{n(n+1)}{2}$ dimensions.

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Among the inequalities (1) can be found dependent inequalities. Let us suppose that one has chosen a system of independent inequalities

$$\rho_1 \geq 0, \rho_2 \geq 0, \ldots, \rho_m \geq 0$$

which define the domain $\Delta$. With the help of independent regulators $\rho_1, \rho_2, \ldots, \rho_m$ one will present all the regulators under the form

$$\rho_k = \sum_{r=1}^{m} h_r^{(k)} \rho_r \text{ where } h_r^{(k)} \geq 0. \ (r = 1, 2, \ldots, m; k = 1, 2, \ldots, \sigma) \qquad (2)$$

Let us observe that any one quadratic form $\sum \sum a_{ij} x_i x_j$ does not verify the equations

$$\rho_1 = 0, \rho_2 = 0, \ldots, \rho_m = 0$$

because the equalities (2) give

$$\rho_k = 0, \quad (k = 1, 2, \ldots, \sigma)$$

and, by virtue of the formula (I) of Number 84, one has

$$\sum \sum a_{ij} a'_{ij} = 0,$$

$\sum \sum a'_{ij} x_i x_j$ being an arbitrary form; it follows that

$$a_{ij} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

To the domain $\Delta$, therefore, the conclusion deduced in my first mémoire cited † are applicable.

Let us indicate by

$$\varphi_1, \varphi_2, \ldots, \varphi_s \tag{3}$$

the quadratic forms which characterise the various edges of the domain $\Delta$.

The domain $\Delta$ of quadratic forms will be determined by the equalities

$$\sum \sum a_{ij} x_i x_j = \sum_{u=1}^{s} u_k \varphi_k \text{ where } u_k \geq 0, \quad (k = 1, 2, \ldots, s)$$

$u_1, u_2, \ldots, u_s$ being positive arbitrary parameters or zeros.

Let us notice that by virtue of the formula (I) of Number (84), each form $\varphi_k (k = 1, 2, \ldots, s)$ of the series (3) will have for expression

$$\varphi_k = \sum_{r=1}^{\sigma} \lambda_r^{(k)} (p_{1r} x_1 + p_{2r} x_2 + \ldots + p_{nr} x_n)^2$$
$$\text{where } \lambda_r^{(4)} \geq 0. \quad (r = 1, 2, \ldots, \sigma; k = 1, 2, \ldots, \sigma)$$

*Properties of independent regulators*

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By keeping the notations from Number 69–74 let us suppose that a simplex $L$ of the set $(L)$ is characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}).$$

Let us suppose that among the regulators

$$\rho_0, \rho_1, \ldots, \rho_n$$

which correspond to the various faces in $n-1$ dimensions of the simplex $L$, is found at least one independent regulator. Let us suppose, to fix the ideas, that $\rho_0$ is a similar regulator.

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† This journal V. 133, p. 97

Let us indicate by $L_k$ $(k = 0, 1, 2, \ldots, n)$ the simplexes which are contiguous to the simplex $(L)$ through the faces in $n-1$ dimensions characterised by the systems

$$(l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = 0, 1, 2, \ldots, n)$$

Let us suppose that by replacing in the simplex $L$ the vertice $(l_{ik})$ by a vertice $(l'_{ik})$, one obtains the simplex $L_k$ $(k = 0, 1, 2, \ldots, n)$.

By virtue of the formula (6) of Number 73, one will have

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = F_{(L)}(l'_{1k}, l'_{2k}, \ldots, l'_{nk}). \quad (k = 0, 1, 2, \ldots, n; h \neq k) \tag{1}$$

Let us admit

$$l'_{ik} = \sum_{k=0}^{n} \vartheta_r^{(k)} l_{ik} \text{ where } \sum_{r=0}^{n} \vartheta_r^{(k)} = 1. \quad (k = 0, 1, 2, \ldots, n) \tag{2}$$

As, because of the inequality (14) of Number 70,

$$\sum p_{ik}(l_{ih} - l'_{ik}) > 0, \quad (h \neq k)$$

it becomes, by virtue of (2)

$$\vartheta_k^{(k)} < 0. \quad (k = 0, 1, 2, \ldots, n) \tag{3}$$

Let us examine the numbers

$$\vartheta_0^0, \vartheta_1^0, \ldots, \vartheta_n^0 \tag{4}$$

which correspond to the independent regulator $\rho_0$. One will have, because of (3),

$$\vartheta_0^0 = 0.$$

I say that among the numbers $\vartheta_1^0, \vartheta_2^0, \ldots, \vartheta_n^0$ at least two numbers are positive. As $\sum_{k=0}^{n} \vartheta_k^0 = 1$, it is evident that at least one number, for example $\vartheta_n^0$, will be positive. Let us suppose that $\vartheta_n^0$ is the only positive number in the series (4).

Let us indicate, to fix the ideas,

$$\vartheta_0^0 < 0, \vartheta_1^0, \ldots, \vartheta_\lambda^0 < 0, \vartheta_{\lambda[+1]}^0 = 0, \ldots, \vartheta_{n-1}^0 = 0, \vartheta_n^0 > 0. \tag{5}$$

The corresponding value of the function $F_{(L)}(l'_{10}, \ldots, l'_{n0})$, by virtue of the formula (4) of Number 78, can be presented under the form

$$F_{(L)}(l'_{10},\ldots,l'_{n0}) = \sum\sum a_{ij}(l'_{i0}-l_{in})(l'_{j0}-l_{jn}) \\ -\sum_{k=0}^{\lambda}\vartheta_k\sum\sum a_{ij}(l_{ik}-l_{in})(l_{jk}-l_{jn}). \tag{6}$$

By virtue of the formula (I) of Number 84 and of inequalities (5), one can present this equality under the form

$$F_{(L)}(l'_{10},\ldots,l'_{n0}) = \sum_{r=1}^{\sigma} h_r\rho_r \text{ where } h_r \geq 0,$$

and as on the other hand, because of (1)

$$2\rho_0\sum p_{i0}(l_{ih}-l'_{i0}) = F_{(L)}(l'_{i0},\ldots,l'_{n0}), \tag{7}$$

it becomes

$$\rho_0 = \sum_{r=1}^{\sigma} g_r\rho_r \text{ where } g_r \geq 0. \quad (r = 1,2,\ldots,\sigma)$$

We have supposed that $\rho_0$ is an independent regulator, therefore it is necessary that

$$g_2 = 0 \text{ so long as a regulator } \rho_r \text{ is not proportional to } \rho_0.$$

The formula (6) gives

$$\sum\sum a_{ij}(l'_{i0}-l_{in})(l'_{j0}-l_{jn}) = \delta\rho_0 \text{ where } \delta > 0,$$
$$\sum\sum a_{ij}(l_{ik}-l_{in})(l_{jk}-l_{jn}) = \delta_k\rho_0 \text{ where } \rho_k > 0.$$
$$(k = 0,1,2,\ldots,\lambda)$$

It follows that one has identically,

$$\sum\sum a_{ij}(l'_{i0}-l_{in})(l'_{j0}-l_{jn}) = \frac{\delta}{\delta_k}\sum\sum a_{ij}(l_{ik}-l_{in})(l_{jk}-l_{jn}).$$

For this identity to hold, it is necessary and sufficient that

$$l'_{i0}-l_{in} = \sqrt{\frac{\delta}{\delta_k}}(l_{ik}-l_{in}).$$

By virtue of Theorem I of Number 51, the numbers $l'_{i0}-l_{in}$ $(i = 1,2,\ldots,n)$, and $l_{ik}-l_{in}$ $(i = 1,2,\ldots,n)$, do not have common divisor, one concludes that

$$l'_{i0} = l_{ik},$$

which is impossible.

Let us indicae, to fix the ideas,

$$\vartheta_0^0 < 0, \vartheta_1^0 < 0, \dots, \vartheta_\lambda^0 < 0, \vartheta_{\lambda+1}^0 = 0, \dots, \vartheta_\mu^0 = 0, \vartheta_{\mu+1}^0 > 0, \dots, \vartheta_n^0 > 0 \tag{8}$$

where $\lambda \geq 0$ and $\mu \leq n-2$.

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*Theorem: By replacing in the simplex $L$ the vertices $(l_{ik}), k = 0, 1, 2, \dots, \lambda$, successively by the vertice $(l'_{i0})$ one will obtain the simplexes*

$$L_0, L_1, \dots, L_\lambda \tag{9}$$

*which are contiguous to the simplex $L$ and one to one by the faces in $n-1$ dimensions the regulators of which are proportional to the regulator $\rho_0$.*

Let us apply the formula $(*)$ of Number 74 to the simplex $L$ and $L_k$ $(k = 0, 1, 2, \dots, \lambda)$; one will have

$$F_{(L)}(l'_{10}, \dots, l'_{n0}) = F_{(L_k)}(l'_{10}, \dots, l'_{n0}) + 2\rho_k \sum p_{ik}(l_{ih} - l'_{i0}).$$
$$(h \neq k; n = 0, 1, 2, \dots, \lambda)$$

By virtue of (8), one obtains,

$$\sum p_{ik}(l_{ih} - l'_{i0}) > 0. \quad (h \neq k; k = 0, 1, 2, \dots, \lambda)$$

In view of (8), one finds

$$F_{(L_k)}(l'_{10}, \dots, l'_{n0}) = \delta_k \rho_0 \text{ where } \delta_k \geq 0 \quad (k = 0, 1, 2, \dots, \lambda)$$

and

$$\rho_k = u_k \rho_0 \text{ where } u_k > 0. \ (k = 0, 1, 2, \dots, \lambda) \tag{10}$$

On the grounds of (1) and (7), it becomes

$$F_{(L)}(l'_{1k}, \dots, l'_{nk}) = \omega_k F_{(L)}(l'_{10}, \dots, l'_{n0}) \text{ where } \omega_k > 0. \ (k = 0, 1, 2, \dots, \lambda)$$

The equality obtained presents an identity with regard to the coefficients of the quadratic form $\sum\sum a_{ij} x_i x_j$. One derives, because of (2), the equalities

$$\left(\vartheta_i^{(k)}\right)^2 - \vartheta_i^{(k)} = \omega_k \left((\vartheta_i^0)^2 - \vartheta_i^0\right), \quad (i = 0, 1, 2, \dots, n)$$
$$\vartheta_i^{(k)} \vartheta_j^{(k)} = \omega_k \vartheta_i^0 \vartheta_j^0. \quad (i = 0, 1, 2, \dots, n; j = 0, 1, 2, \dots, n; i \neq j)$$

As $\sum_{i=0}^{n}\vartheta_i^{(k)} = 1$ and $\sum_{i=0}^{n}\vartheta_i^0 = 1$, it is necessary that $\omega_k = 1$ and

$$\vartheta_i^{(k)} = \vartheta_i^0, \quad (i = 0, 1, 2, \ldots, n)$$

therefore

$$l'_{ik} = l'_{i0}. \quad (k = 0, 1, 2, \ldots, \lambda)$$

The formula (1) becomes in this case

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{i0}) = F_{(L)}(l'_{10}, \ldots, l'_{n0}). \quad (h \neq k; k = 0, 1, 2, \ldots, \lambda)$$

Let us notice that the simplexes

$$L, L_0, L_1, \ldots, L_\lambda$$

make up a group of perfectly determined simplexes corresponding to the independent regulator $\rho_0$. That which we have mentioned concerning the simplex $L$ can be related back to all the simplexes of the series (9). All the simplexes which remain $L_{\lambda+1}, \ldots, L_n$ are contiguous to the simplex $L$ through the faces the regulators of which are not proportional to $\rho_0$.

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Let us notice that the simplexes (9) make up a convex polyhedron $K$ having $n + 2$ vertices

$$(l'_{i0}), (l_{i0}), \ldots, (l_{in}).$$

In effect, all the points of simplexes (9) belong to polyhedron $K$ made up of points determined by the equalities

$$x_i = ul'_{i0} + \sum_{k=0}^{n} u_k l_{ik} \text{ where } u + \sum_{u=0}^{n} u_k = 1 \text{ and } u \geq 0, u_k \geq 0. \quad (11)$$
$$(k = 0, 1, 2, \ldots, n)$$

I argue that any point $(x_i)$ determined by these equalities belongs to at least one of simplex (9).

Let us suppose, in the first place, that one has the inequalities

$$u_k + u\vartheta_k^0 \geq 0. \quad (k = 0, 1, 2, \ldots, n)$$

One will present the equation (11), because of (2), under the following form:

$$x_i = \sum_{k=0}^{n}(u_k + u\vartheta_k^0) l_{ik}$$

and, as $\sum_{k=0}^{n}(u_k + u\vartheta_k^0) = 1$, one concludes that the point $(x_i)$ belongs to the simplex $L$.

This laid down, let us suppose that at least one of numbers $u_k + u\vartheta_k^0$, $k = 0, 1, 2, \dots, n$, is negative. Let us choose among the numbers

$$\frac{u_0}{\vartheta_0^0}, \frac{u_1}{\vartheta_1^0}, \dots, \frac{u_\lambda}{\vartheta_\lambda^0}$$

which are all negatives or zeros, because of (8) and (11), a number $\frac{u_k}{\vartheta_\lambda^0}$ the numerical value of which is the smallest. The point $(x_i)$ determined by the equalities (11) belongs in this case to the simplex $L_k$. To demonstrate this, one will present the equalities (11) under the form

$$x_i = \left(u + \frac{u_k}{\vartheta_k}\right) l'_{i0} + \sum_r \left(u_r - u_k \frac{\vartheta_r}{\vartheta_k}\right) l_{ir}. \quad (r = 0, 1, 2, \dots, n; r \neq k)$$

On the ground of suppositions made, one will have the inequalities

$$u + \frac{u_k}{\vartheta_k} > 0, \quad u_r - u_k \frac{\vartheta_r}{\vartheta_k} \geq 0, \quad (r = 0, 1, 2, \dots, n; r \neq k)$$

and, as

$$u + \frac{u_k}{\vartheta_k} + \sum_r \left(u_r - u_k \frac{\vartheta_r}{\vartheta_k}\right) = 1,$$

one concludes that the point $(x_i)$ belongs to the simplex $(L_k)$ $(u = 0, 1, 2, \dots, \lambda)$.

Let us examine the faces in $n-1$ dimensions of the polyhedron $K$. On the ground of conditions (8), the polyhedron $K$ possesses $\mu - \lambda$ faces in $n-1$ dimensions $Q_k$ which are characterised by $n+1$ vertices

$$(l'_{i0}), (l_{ih}). \quad (h = 0, 1, 2, \dots, n; h \neq k; k = \lambda + 1, \dots, \mu)$$

The vertice $(l_{ik})$ where $k = \lambda + 1, \dots, \mu$ is opposite to the face $\theta_k (k = \lambda + 1, \dots, \mu)$.

All the faces in $n-1$ dimensions of the polyhedron $K$ which remain are characterised by $n$ vertices. One will characterise them in the polyhedron $K$ by two opposite vertices.

One obtains in this way $n - \mu$ faces $P_k$ $(k = \mu + 1, \dots, n)$ of the polyhedron $K$ characterised by two opposite vertices $(l'_{i0})$ and $l_{ik}$ $(k = \mu + 1, \dots, n)$ and one obtains $(\lambda + 1)(n - \mu)$ faces $P_{kh}(h = 0, 1, 2, \dots, \lambda; k = \mu + 1, \dots, n)$ characterised by two opposite vertices $l_{ik}$ and $l_{ih}$.

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Let us notice that the polyhedron $K$ is contiguous through the faces $Q_k$ $(k = \lambda + 1, \ldots, \mu)$ to other independent regulator $\rho_0$.

to demonstrate this, let us examine the simplex $L_k$ $(k = \lambda + 1, \ldots, \mu)$ contiguous to the simplex $L$ through the face in $n-1$ dimensions characterised by the vertices

$$(l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

This face presents a part of the corresponding face $Q_k$ of the polyhedron $K$.

By applying the formula $(*)$ of Number 74 to the simplexes $L$ and $L_k$, one obtains

$$F_{(L)}(l'_{10}, \ldots, l'_{n0}) = F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) + 2\rho_k \sum p_{ik}(l_{ih} - l'_{i0})$$

$$\text{where } h \neq k \text{ and } n = \lambda + 1, \ldots, \mu.$$

On the ground of conditions (8), one will have

$$\sum p_{ik}(l_{ih} - l'_{i0}) = 0, \quad (k = \lambda + 1, \ldots, \mu)$$

therefore, because of (7),

$$F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) = F_{(L)}(l'_{10}, \ldots, l'_{n0}) = 2\rho_0 \sum p_{i0}(l_{ih} - l'_{i0}).$$
$$(h \neq 0; k = \lambda + 1, \ldots, \mu)$$

As the point $(l'_{10})$ is not found among the vertices of the simplex $L_k$, it is necessary that among the regulators of faces of the simplex $L_k$ are found, by virtue of the equation obtained, regulators which are proportional to $\rho_0$.

By noticing that

$$l'_{i0} = \sum \vartheta^0_h l_{ih} + \vartheta^0_k l'_{ik}, \quad \text{where } \sum \vartheta^0_h + \vartheta^0_k = 1$$
$$(h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

since because of (8), $\vartheta^0_k = 0$ $(k = \lambda + 1, \ldots, \mu)$, one concludes, on the ground of the previous theorem, that by replacing in the simplex $L_k$ the vertices $(l_{ih})$ $(h = 0, 1, 2, \ldots, \lambda)$ by the vertice $(l'_{i0})$ one will obtain a group of simplexes

$$L_k, L_k^{(0)}, L_k^{(1)}, \ldots, L_k^{(\lambda)}, \quad (k = \lambda + 1, \ldots, \mu) \tag{12}$$

which are contiguous one to one by faces in $n-1$ dimensions the regulators of which are proportional to $\rho_0$.

Let us indicate by $K_h$ the convex polyhedron made up of simplexes (12). One obtains the polyhedron $K_k$ by replacing in the polyhedron $K$ the vertice $(l_{ik})$ by the vertice $l'_{ik}$ $(k = \lambda + 1, \dots, \mu)$. One concludes that the polyhedra $K$ and $K_k$ are contiguous through the face $Q_k$.

Let us examine other faces of the polyhedra $K$. The face $P_k$ $(k = \mu + 1, \dots, n)$ belongs to the simplex $L$ which is contiguous through the face $P_k$ to the simplex $L_k$. The regulator $\rho_k$ $(k = \mu + 1, \dots, n)$ of this face can not be proportional to the independent regulator $\rho_0$.

It may turn out that any one of regulators corresponding to the various faces of the simplex $L_k$ is not proportional to the regulator $\rho_0$. In this case, the polyhedron $K$ will not be contiguous through the face $P_k$ to any one analogous polyhedron corresponding to the independent regulator $\rho_0$.

It may also turn out that among the regulators of faces of the simplex $L_k$ are found regulators which are proportional to $\rho_0$; in this case, the simplex $L_k$ belongs to a convex polyhedron $K_k$ which is contiguous to $K$ through the face $P_k$ $(k = \mu + 1, \dots, n)$.

In the same way, one will examine the analogous faces $P_{hk}$ of the polyhedron $K$ $(h = 0, 1, 2, \dots, \lambda; k = \mu + 1, \dots, n)$.

By applying the procedure shown to the various incongruent simplexes of the set $(L)$, one will determine the incongruent convex polyhedron

$$K, K_1, \dots, K_{\omega-1}$$

which are made up of corresponding groups of simplexes belonging to the set $(L)$.

*Reconstruction of the set $(L)$ of simplexes by another set $(L')$ of simplexes.*

91

One can partition the convex polyhedra

$$K, K_1, \dots, K_{\omega-1} \tag{1}$$

corresponding to an independent regulator $\rho$ into new simplexes.

By keeping the previous notations, let us examine the convex polyhedron $K$ made up of simplexes

$$L, L_0, \dots, L_\lambda. \tag{2}$$

The polyhedron $K$ possesses $n + 2$ vertices

$$(l'_i), (l_{i0}), \dots, (l_{in}).$$

it Theorem. By replacing in the simplex $L$ characterised by the vertices

$$(l_{i0}), (l_{i1}), \dots, (l_{in})$$

a value $(l_{ik})$ by the vertice $(l'_i)$ where $k = \mu + 1, \dots, n$, one obtains $n - \mu$ simplexes

$$L_{\mu+1}, \dots, L'_n \tag{3}$$

which also make up the polyhedron $K$. The simplexes obtained do not belong to the set $(L)$ of simplexes.

Let us write, as we have done in Number 87,

$$l'_i = \sum_{k=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \tag{4}$$

and

$$\vartheta_0 < 0, \vartheta_1 < 0, \dots, \vartheta_\lambda < 0, \vartheta_{\lambda+1} = 0, \dots, \vartheta_\mu = 0, \vartheta_{\mu+1} > 0, \dots, \vartheta_n > 0. \tag{5}$$

It is clear that each point of simplex (3) belongs to the polyhedron $K$.

Let $(x_i)$ be any one point of the polyhedron $K$ determined with the help of equalities

$$x_i = ul'_i + \sum_{k=0}^{n} u_k l_{ik} \text{ where } u + \sum u_k = 1, u \geq 0, u_k \geq 0. \ (k = 0, 1, \dots, n) \tag{6}$$

Let us choose among the numbers

$$\frac{u_{\mu+1}}{\vartheta_{\mu+1}}, \dots, \frac{u_n}{\vartheta_n}$$

the one which is the smallest. Let us suppose, to fix the ideas, that

$$\frac{u_r}{\vartheta_r} \geq \frac{u_k}{\vartheta_k}. \quad (r = \mu + 1, \dots, n)$$

I argue that the point $(x_i)$ belongs to the simplex $L'_k$. In effect, the equality (6) can be speculated, because of (4), under the form

$$x_i = \left(u + \frac{u_k}{\vartheta_k}\right) l'_i + \sum_h \left(u_h - u_k \frac{\vartheta_h}{\vartheta_k}\right) l_{ih}. \quad (h = 0, 1, 2, \dots, n; h \neq k)$$

By observing that

$$u + \frac{u_k}{\vartheta_k} \geq 0, \; u_h - u_k \frac{\vartheta_h}{\vartheta_k} \geq 0, \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

and that

$$u + \frac{u_k}{\vartheta_k} + \sum_h \left( u_k - u_k \frac{\vartheta_h}{\vartheta_k} \right) = 1 \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

one concludes that the point $(x_i)$ belongs to the simplex $L'_k$ $(k = \mu + 1, \ldots, n)$.

The simplexes (3) can not belong to the set $(L)$, because this set, by virtue of Theorem I of Number 61, uniformly partition the space in $n$ dimensions.

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Let us suppose that one has replaced in the set $(L)$ the group of simplexes (2) by the corresponding group (3). Let us suppose that one has effected this reconstruction of simplexes of the set $(L)$ with regard to all the polyhedra which are congruent to the polyhedra (1). One obtains in this way a new set $(L')$ of simplexes which enjoy the following properties.

*1. The set* $(L')$ *of simplexes uniformly fills the space in* $n$ *dimensions.*

*2. The set* $(L')$ *can be divided into classes of congruent simplexes and the number of different classes is finite.*

Let us find the regulators and the characteristics of faces in $n-1$ dimensions of simplexes belonging to the set $(L')$.

Let $L'$ and $L'_0$ be any two simplexes of the set $(L')$ which are contiguous through a face $P$ in $n-1$ dimensions. Suppose that the two simplexes $L'$ and $L'_0$ also belong to the set $(L')$. In this case the regulator and the characteristic of the face $P$ in the set $(L')$ do not change.

Let us suppose that at least one of the simplexes examined does not belong to the set $(L)$ of simplexes. This simplex will belong in this case to a polyhedron which is congruent to a polyhedron of the series (1). Let us suppose to fix the ideas that this is the polyhedron $K$.

By noticing that the simplex examined is found among the simplexes (3) let us choose one of these simplexes $L'_k$ $(k = \mu + 1, \ldots, n)$ and examine the regulators and the characteristics of all these faces in $n-1$ dimensions.

By virtue of the definition established, the simplex $L'_k$ is characterised by the vertices

$$(l_{i0}), \dots, (l_{i,k-1}), (l'_i), (l_{i,k+1}), \dots, (l_{in}). \quad (k = \mu + 1, \dots, n)$$

Let us indicate by $P_{hk}$ a face in $n-1$ dimensions of the simplex $L'_k$ which is opposite to the vertice $(l_{ih})$ $(h = 0, 1, 2, \dots; h \neq k)$. By $P'_k$ let us indicate the faces of simplex $L'_k$ which is opposite to the vertice $l'_i$.

All the faces in $n-1$ dimensions of the simplex $L'_k$ can be divided into three groups:

1. $P_{0k}, P_{1k}, \dots, P_{\lambda k}$ and $P_k$;
2. $P_{\lambda+1,k}, \dots, P_{\mu k}$;
3. $P_{\mu+1,k}, \dots, P_{k-1,k}, P_{k+1,k}, \dots, P_{nk}$.

93 Let us find the regulators of faces of the simplex $L'_k$ belonging to the first group.

Let us examine, in the first place, the face $P_k$. As the face $P_k$ is characterised by the vertices

$$(l_{i0}), \dots, (l_{i,k-1}), (l_{i,k+1}), \dots, (l_{in}),$$

it presents a face of the polyhedron $K$.

In the set $(L)$ the face $P_k$ would belong to two simplexes $L$ and $L_k$. Two cases to distinguish:

*First case:* the simplex $L_k$ belongs to the set $(L')$.

Let us indicate by $\rho_k$ the regulator and by $(P_{ik})$ the characteristic of the face $P_k$ in the set $(L)$ with regard to the simplex $L$.

Let us indicate by $\rho'_k$ the regulator of the face $P_k$ in the set $(L')$. The characteristics of the face $P_k$ in the set $(L')$ with regard to the simplex $L'_k$ will be $(P_{ik})$.

By virtue of the definition established in Number 73, one can declare

$$F_{(L_k)}(l'_1, \dots, l'_n) = 2\rho'_k \sum (-p_{ik})(l_{ih} - l'_i). \quad (h \neq k)$$

By applying the formula $(*)$ of Number 74 to the simplex $L$ and $L_k$, one obtains

$$F_{(L)}(l'_1, \dots, l'_n) = F_{(L_k)}(l'_1, \dots, l'_n) + 2\rho_k \sum p_{ik}(l_{ih} - l'_i). \quad (h \neq k) \tag{7}$$

It follows that

$$\rho'_k = \rho_k + \frac{F_{(L)}(l'_1, \dots, l'_n)}{2 \sum p_{ik}(-l_{ih} + l'_i)}. \quad (k = \mu + 1, \dots, n) \tag{8}$$

We have seen in Number 87 that the function $F_{(L)}(l'_1,\dots,l'_n)$ is proportional to the independent regulator $\rho$. As, by virtue of (5),

$$\sum p_{ik}(-l_{ih}+l'_i)>0, \quad (h\neq k; k=\lambda+1,\dots,\mu)$$

the formula (8) can be written

$$\rho'_k=\rho_k+\delta_k\rho \ \text{ where } \ \delta_k>0.$$

*Second case*: the simplex $L_k$ does not belong to the set $(L')$.

In this case the simplex $L_k$ belongs to a convex polyhedron $K_k$ and the face $P_k$ will belong in the set $(L')$ to two new simplexes.

The face $P_k$ in the set $(L')$ belongs to the simplex $L'_k$ and to a simplex which one obtains by replacing the vertice $l'_{ik}$ of the simplex $L_k$ by a new vertice which one will indicate by $(l^0_i)$. Let us indicae by $L^0_k$ the simplex which one obtains by replacing in the simplex $L_k$ the vertice $l'_{ik}$ by the vertice $(l^0_i)$.

By virtue of the definition established in Number 73, one will have

$$F_{(L'_k)}(l^0_1,\dots,l^0_n)=2\rho'_k\sum p_{ik}(l_{ih}-l^0_i). \quad (h\neq k)$$

By applying the fundamental formula $(*)$ of Number 74 to the simplexes $L_k$ and $L'_k$ which are contiguous through the face $P_k$, one obtains

$$F_{(L_k)}(l^0_1,\dots,l^0_n)=F_{(L'_k)}(l^0_1,\dots,l^0_n)+\sum(-p_{ik})(l_{ih}-l^0_i)\frac{F_{(L_k)}(l'_1,\dots,l'_n)}{\sum(-p_{ik})(l_{ih}-l'_i)}$$

and, because of (7), it becomes

$$\rho'_k=\rho_k+\frac{F_{(L)}(l'_1,\dots,l'_n)}{2\sum p_{ik}(-l_{ih}+l'_i)}+\frac{F_{(L_k)}(l^0_1,\dots,l^0_n)}{2\sum p_{ik}(l_{ih}-l^0_i)}. \tag{9}$$

On the ground of the supposition made, the functions

$$F_{(L)}(l'_1,\dots,l'_n) \ \text{ and } \ F_{(L_k)}(l^0_1,\dots,l^0_n)$$

are proportional to the independent regulator $\rho$. As

$$\sum p_{ik}(-l_{ih}+l'_i)>0 \ \text{ and } \ \sum p_{ik}(l_{ih}-l^0_i)>0, \quad (k=\mu+1,\dots,n)$$

the formula (9) can be written

$$\rho'_k=\rho_k+\delta_k\rho \ \text{ where } \ \delta_k>0. \quad (k=\mu+1,\dots,n)$$

In the same way, one will examine the regulator of the face $P_{hk}; (h=0,1,2,\dots,\lambda)$.

As the face $P_{hk}$ belongs in the set $(L)$ to the simplex $L_h$ $(h = 0, 1, 2, \dots, \lambda)$, one concludes that by designating the simplex $L_h$ with the simplex $L$ one will return to one of the two previous cases.

94 Let us find the regulators of faces of the simplex $L'_k$ belonging to the second group. Choose one face $P_{hk}$ $(h = \lambda + 1, \dots, \mu; k = \mu + 1, \dots, n)$ in this group.

The face $P_{hk}$ presents a part of the face $Q_k$ of the polyhedron $K$.

We have indicated in Number 90 by $K_h$ the polyhedron which is contiguous to the polyhedron $K$ through the face $Q_h$. The polyhedron $K_h$ posses $n + 1$ vertices

$$(l'_i), (l_{i0}), \dots, (l_{i,h-1}), (l'_{ih}), (l_{i,h+1}), \dots, (l_{in}).$$

In the set $(L')$, the polyhedron $K_h$ is partitioned into simplexes

$$L'_{\mu+1,h}, \dots, l'_{n,h}$$

which one obtains by replacing in the simplexes

$$L'_{\mu+1}, \dots, L'_n$$

the vertice $(l_{ih})$ by the vertice $(l'_{ih})$ $(h = \lambda + 1, \dots, \mu)$.

One concludes that the face $P_{hk}$ belongs to the set $(L')$ to two simplexes

$$L'_k \quad \text{and} \quad L'_{kh}.$$

Let us indicate by $\rho'_{hk}$ the regulator corresponding to the face $P_{hk}$ in the set $(L')$. Notice that the characteristic $(P'_{ih})$ will be the same for all the faces $P_{hk}$ where $k = \mu + 1, \dots, n$ because these faces make up the face $Q_k$ of the polyhedron $K$.

In the set $(L)$, the face $Q_h$ is partitioned into faces

$$P_h, P_{h0}, \dots, P_{h\lambda}$$

of simplexes $L, L_0, \dots, L_\lambda$ which have the same characteristic $(p_{ih})$. One concludes that

$$p'_{ih} = p_{ih},$$

provided that the characteristic $(p_{ih})$ is chosen with regard to the simplexes (2).

By virtue of the definition established in Number 73, one can write

$$F_{(L_k)}(l'_{ih}, \dots, L'_{nh}) = 2\rho'_{hk} \sum p_{ih}(l_{ir} - l'_{ih}). \quad (r \neq h)$$

By applying the formula $(*)$ of Number 74 to the simplexes $L$ and $L'_k$, one will have

$$F_{(L)}(l'_{ih},\ldots,L'_{nh}) = F_{(L'_k)}(l'_{ih},\ldots,L'_{nh}) + \sum p_{ik}(l_{ir} - l'_{ih})\frac{F_{(L)}(l'_1,\ldots,l'_n)}{\sum p_{ik}(l_{ir} - l'_i)}.$$

Besides, one has

$$F_{(L)}(l'_{ih},\ldots,L'_{nh}) = 2\rho_h \sum p_{ih}(l_{ir} - l'_{ih}), \quad (r \neq h)$$

and consequently

$$\rho'_{hk} = \rho_h + \frac{\sum p_{ik}(l_{ir} - l'_{ih})}{\sum p_{ih}(l_{ir} - l'_{ih})} \cdot \frac{F_{(L)}(l'_1,\ldots,l'_n)}{2\sum(p_{ik} - l_{ir} + l'_i)}.$$

One can thus write

$$\rho'_{hk} = \rho_h + \delta_{hk}\rho. \quad (h = \lambda+1,\ldots,\mu; k = \mu+1,\ldots,n)$$

In the formula obtained, the number $\delta_{hk}$ can be positive, negative or zero.

95

Let us find the regulators of faces of the simplex $L'_k$ belonging to the third group. Let $P_{hk}$ be a face belonging to this group, $h = \mu+1,\ldots,n$, $h \neq k$, $k = \mu+1,\ldots,n$. The face $P_{hk}$ belongs in the set $(L')$ to two simplexes $L'_k$ and $L'_h$ of the series (3). By replacing in the simplex $L'_k$ the vertices $(l_{ih})$ by the vertice $(l_{ik})$, one obtains the simplex $L'_h$. This results in that by indicating with $\rho_{hk}$ the regulator and with $(p_i^{(hk)})$ the characteristic of the face $P_{hk}$ with respect to the simplex $L'_k$, one will have

$$F_{(L'_k)}(l_{1k},\ldots,l_{nk}) = 2\rho'_{hk}\sum p_i^{(hk)}(l_{ir} - l_{ik}). \quad (r \neq h; r \neq k) \tag{10}$$

The equality (4) can be written

$$l_{ik} = \frac{1}{\vartheta_k}l'_i + \sum_r \left(-\frac{\vartheta_r}{\vartheta_k}\right) l_{ir}. \quad (r = 0,1,2,\ldots,n; r \neq k)$$

By noticing that

$$\frac{1}{\vartheta_k} + \sum_r \left(-\frac{\vartheta_r}{\vartheta_k}\right) = 1,$$

one will determine the value of the function $F_{(L'_k)}(l_{1k},\dots,l_{nk})$, by virtue of the formula (4) of Number 73, by the equality

$$F_{(L'_k)}(l_{1k},\dots,l_{nk}) = \sum\sum a_{ij}l_{ik}l_{jk} - \frac{1}{\vartheta_k}\sum\sum a_{ij}l'_il'_j + \sum_r \frac{\vartheta_r}{\vartheta_k}\sum\sum a_{ij}l_{ir}l_{jr}. \quad (r = 0,1,2,\dots,n; r \neq k)$$

By recalling that because of (4)

$$F_{(L)}(l'_1,\dots,l'_n) = \sum\sum a_{ij}l'_il'_j - \sum_{k=0}^{n}\vartheta_k\sum\sum a_{ij}l_{ik}l_{jk},$$

and by comparing the two equalities obtained, one finds

$$F_{(L'_k)}(l_{1k},\dots,l_{nk}) = -\frac{1}{\vartheta_k}F_{(L)}(l'_1,\dots,l'_n).$$

By substituting in the formula (10) the expression found of the function $F_{(L'_k)}(l_{1k},\dots,l_{nk})$, one obtains

$$\rho'_{hk} = -\frac{1}{\vartheta_k}\cdot\frac{F_{(L)}(l'_1,\dots,l'_n)}{2\sum p_i^{(hk)}(l_{ir}-l_{ik})}.$$
$$(h = \mu+1,\dots,n; k = \mu+1,\dots,n; h \neq k; r \neq k)$$

One concludes that by admitting

$$\rho'_{hk} = -\delta_{hk}\rho, \quad (h = \mu+1,\dots,n; k = \mu+1,\dots,n; h \neq k)$$

one will have $\delta_{hk} > 0$.

*Algorithm for the study of domains of quadratic forms which are contiguous to a given domain through the faces in* $\frac{n(n+1)}{2} - 1$ *dimensions.*

96

Let us suppose that a domain $\Delta$ of quadratic form corresponding to a type of primitive parallelohedra which is characterised by the set $(L)$ of simplexes is defined by the independent inequalities

$$\rho_k \geq 0. \quad (k = 1,2,\dots,m)$$

Let us suppose that one of these regulators is proportional to an independent regulator $\rho$ and construct the set $(L)$ of simplexes in another set $(L')$ with the help of the procedure shown in Number 91–92.

Let us indicate by

$$\rho_1, \rho_2, \dots, \rho_\sigma$$

all the regulators of incongruent faces of simplexes belonging to the set $(L)$ and indicate by

$$\rho'_1, \rho'_2, \dots, \rho'_\tau$$

all the regulators of faces of simplexes belonging to the set $(L')$.

We have seen in Number 93–95 that all these regulators can be presented under the form

$$\begin{cases} \text{either} \quad \rho'_k = -\delta_k \rho \quad \text{where } \delta_k > 0, \\ \text{or} \quad \rho'_k = \rho_k + \delta_{kh}\rho \end{cases} \tag{1}$$

so long as a regulator $\rho'_k$ is not proportional to $\rho$.

Let us examine the domain $D'$ of quadratic forms determined by the inequalities

$$\rho'_k \geq 0. \quad (k = 1, 2, \dots, \tau) \tag{2}$$

I argue that these inequalities define a domain of quadratic forms in $\frac{n(n+1)}{2}$ dimensions. By supposing the contrary, one will find the parameters $u_k$ $(k = 1, 2, \dots, \tau)$ positive or zero which reduce into an identity the equality

$$\sum_{k=1}^{\tau} u_k \rho'_k = 0 \text{ where } u_k \geq 0. \ (k = 1, 2, \dots, \tau) \tag{3}$$

By virtue of formulae (1), this identity can be written

$$\sum_{k=1}^{\sigma} v_k \rho_k \pm v\rho = 0 \text{ where } v_k \geq 0. \ (k = 1, 2, \dots, \sigma)$$

As the regulator $\rho$ is independent it is necessary that $v_k = 0$ as long as a regulator $\rho_k$ is not proportional to $\rho$. This results in that within the identity (3) one also has $u_k = 0$ so long as a regulator $\rho'_k$ is not proportional to $\rho$. By virtue of (1), the identity (3) takes the form

$$\sum u_k(-\delta_k \rho) = 0 \text{ where } u_k \geq 0 \text{ and } \delta_k > 0,$$

which is impossible.

The domain $\Delta'$ defined by the inequalities (2) corresponds to a new type of primitive parallelohedra characterised by the set $(L')$ of simplexes.

Let us notice that by virtue of inequalities (1), any quadratic form which is interior to the face of the domain $\Delta$ determined by the equation

$$\rho = 0 \tag{4}$$

belongs to the domain $\Delta'$ and vice-versa. One concludes that the two domain $\Delta$ and $\Delta'$ are contiguous through the face in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation (4).

*Set* $(\Delta)$ *of domains of quadratic forms corresponding to the different types of primitive parallelohedra.*

97

With the help of the algorithm explained in the previous Number, one can determine the domains of quadratic forms

$$\Delta_1, \Delta_2, \ldots, \Delta_m \tag{1}$$

which are contiguous to the domain $\Delta$ by their faces in $\frac{n(n+1)}{2} - 1$ dimensions, then one will determine the domains which are contiguous to the domains (1) and so on.

Let us indicate by $(\Delta)$ the set composed of all the domains of quadratic terms which correspond to the various types of primitive parallelohedra.

*Theorem I. The set* $(\Delta)$ *of domains of quadratic forms uniformly divides the set of all the positive quadratic forms in* $n$ *variables.*

Let $\varphi(x_1, x_2, \ldots, x_n)$ be an arbitrary positive quadratic form. Let us choose a form $\varphi_0(x_1, x_2, \ldots, x_n)$ which is interior to the domain $\Delta$ and let us examine a vector $g$ made up of forms

$$f = \varphi_0 + u(\varphi - \varphi_0) \text{ where } 0 \leq u \leq 1. \tag{2}$$

By making the parameter $u$ grow in a continuous manner in the interval $0 \leq u \leq 1$, one will determine a series of domains

$$\Delta, \Delta', \ldots, \Delta^{(k)} \tag{3}$$

which are successively contiguous through the faces in $\frac{n(n+1)}{2} - 1$ dimensions and which contain the various forms of the vector $g$.

I argue that the series of domains (3) will always be terminated by a domain to which belong the given quadratic form $\varphi$.

To demonstrate this, let us indicate by

$$(l_{i1}), (l_{i2}), \ldots, (l_{i\tau}) \text{ where } \tau = 2^n - 1 \tag{4}$$

the systems which characterise the faces in $n - 1$ dimensions of primitive parallelohedra belonging to the type which corresponds to the domain $\Delta$ of quadratic form.

Let us indicate by the symbol $N(f)$ a sum

$$N(f) = \sum_{h=1}^{\tau} f(l_{1h}, \ldots, l_{nh})$$

of values of a form $f(x_1, x_2, \ldots, x_n)$ corresponding to the systems (4).

Let us indicate, similarly, by

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{i\tau}^{(k)}) \text{ where } \tau = 2^n - 1 \tag{5}$$

the systems which characterise the faces in $n-1$ dimensions of primitive parallelohedra belonging to the type which corresponds to a domain $\Delta^{(k)}$ of the series (3) and declare

$$N^{(k)}(f) = \sum_{h=1}^{\tau} f(l_{1h}^{(k)}, \ldots, l_{nh}^{(k)}). \quad (k = 1, 2, \ldots)$$

We have seen in Number 95 that the systems (4) and (5) are congruent with respect to the modulus 2. By virtue of the theorem of Number 48, one will have an inequality

$$N(f) < N^{(k)}f, \quad (k = 1, 2, \ldots)$$

as long as a quadratic form $f$ is interior to the domain $\Delta$. This results in that the inequality

$$N(f) \leq N^{(k)}(f) \quad (k = 1, 2, \ldots)$$

holds providing that a form $f$ belongs to the domain $\Delta$.

This stated, let us notice that by virtue of the supposition made, the form $\varphi_0$ is interior to the domain $\Delta$, therefore one will have the inequality

$$N(\varphi_0) < N^{(k)}(\varphi_0). \ (k = 1, 2, \ldots) \tag{6}$$

Let $f$ be a form of the vector $g$ which belongs to the domain $\Delta^{(k)}$ of te series (3).

One will have an inequality

$$N(f) \geq N^{(k)}(f). \tag{7}$$

By noticing that because of (2)

$$N(f) = (1-u)N(\varphi_0) + uN(\varphi),$$
$$N^{(k)}(f) = (1-u)N^{(k)}(\varphi_0) + uN^{(k)}(\varphi),$$

the inequality (7) can be written

$$u\left[N(\varphi)-N^{(k)}(\varphi)\right] \geq (1-u)\left[N^{(k)}(\varphi_0)-N(\varphi_0)\right].$$

As $0 < u \leq 1$, this inequality gives, because of (6),

$$N^{(k)}(\varphi) \leq N(\varphi).$$

The quadratic form $\varphi$ being positive, there exist only a limited number of different systems (5) verifying this inequality. Besides, there exist only a limited number of domains of forms belonging to the set $(\Delta)$ which are characterised by the same systems (5). One concludes this that the series (3) will always be terminated by a domain to which belong the given quadratic form $\varphi$.

Let us notice that a quadratic form $\varphi$ which is interior to a domain $\Delta$ does not belong to any other domain of the set $(\Delta)$, since the primitive parallelohedron corresponding to the quadratic form $\varphi$ will belong to the type characterised by the domain $(\Delta)$ and can not belong to any other type of parallelohedra.

Suppose that a positive quadratic form $\varphi$ is interior to a face $P$ in a certain number of dimensions of the domain $\Delta$. The set of all the quadratic forms belonging to the face $P$ will be perfectly determined by a certain type of nonprimitive parallelohedra. One concludes that the form $\varphi$ can not belong to the domains which are contiguous through the face $P$.

98

By effecting the various transformation of the set $(\Delta)$ of quadratic forms with the help of substitutions of integral coefficients and of the determinant which is equal to $\pm 1$, one will do only the permutation of domains of the set $(\Delta)$.

One concludes that the set $(\Delta)$ of domains of forms can be divided into classes of domains composed of equivalent domains.

*Theorem II. The number of various classes of domains belonging to the set* $(\Delta)$ *is finite.*

Let us choose any one domain $\Delta$ of the set $(\Delta)$ and let $\varphi$ be a form which is interior to the domain $\Delta$. We have seen in Number 54 that the positive quadratic form can be transformed into another equivalent form $\varphi'$ which enjoys the property that the system (4) corresponding to the form $\varphi'$ are made up of integers which do not exceed in numerical value a fixed limit.

The form $\varphi'$ is interior to a domain $\Delta'$ which is equivalent to the domain $\Delta$.

As the domain $\Delta'$ is characterised by the systems of integers which do not exceed in numerical value a fixed limit, there exist only a limited number of identical domains in the set $(\Delta)$.

99

With the help of the algorithm introduced in Number 96, one can successively determine all the representatives

$$\Delta, \Delta_1, \ldots, \Delta_{\mu-1} \tag{8}$$

of different classes of domains belonging to the set $(\Delta)$.

The domains obtained enjoy the same property as the domains of quadratic forms which have been studied in my first mémoire cited. † It results in that the domains (8) can serve in the reduction of positive quadratic forms. By calling reduced the positive quadratic forms which belong to the domains (8), one obtains a new reduction method of positive quadratic forms which is entirely analogous to a reduction method of positive quadratic forms introduced in the cited mémoire.

On the nonprimitive parallelohedra corresponding to positive quadratic forms.

100

Let us suppose that a positive quadratic form $\varphi$ defines a primitive parallelohedron $R$.

By virtue of Theorem I of Number 97, the form $\varphi$ belongs at least to the domain of the set $(\Delta)$. The form $\varphi$ can not be interior to any one domain of the set $(\Delta)$ because otherwise the parallelohedron $R$ would be primitive.

Therefore the form $\varphi$ belongs to one face of domains of the set $(\Delta)$.

It results in that the coefficients of the form $\varphi$ verify one of many linear equations

$$\sum\sum p_{ij} a_{ij} = 0$$

to rational coefficients $p_{ij}$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$

One concludes that a positive quadratic form $\sum\sum a_{ij} x_i x_j$ the coefficients of which present an irreducible basis can define only one primitive parallelohedron.

Let us suppose that the examined form $\varphi$ is interior to a face $P$ to any one number of dimensions of domains belonging to the set $(\Delta)$.

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† This Journal, V. 133

Let us indicate by

$$\Delta, \Delta', \ldots, \Delta^{(m)}$$

the domains of the set $(\Delta)$ which are contiguous through the face $P$.

By virtue of that which has been stated in Number 97, one will have the equalities

$$N(\varphi) = N'(\varphi) = \ldots = N^{(m)}(\varphi).$$

One concludes that a positive quadratic form $\varphi$ can belong to only a finite number of domains of the set $(\Delta)$.

101

Let us suppose that an infinite series of quadratic forms

$$f_1, f_2, \ldots \tag{1}$$

is made up of forms which are interior to the domain $\Delta$. Suppose that the forms of this series tend towards a limit $\varphi$.

The forms (1) define an infinite series of primitive parallelohedra

$$R_1, R_2, \ldots$$

belonging to the one type characterised by the domain $\Delta$ which tend towards a limit $R$.

One concludes that any nonprimitive parallelohedron $R$ corresponding to a positive quadratic form $\varphi$ can be considered as a limit of primitive parallelohedra (2).

Let us indicate by the symbol $S_\nu$ the number of faces in $\nu$ dimensions of the nonprimitive parallelohedron $R$ and by $S_\nu^0$ let us indicate the number of faces in $\nu$ dimensions of primitive parallelohedron (2) $(\nu = 0, 1, 2, \ldots, n-1)$.

As the faces of the nonprimitive parallelohedron $R$ are made up of boundaries of faces of primitive parallelohedra belonging to the series (2), one concludes that

$$S_\nu \leq S_\nu^0. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

We have seen in Number 65 that

$$S_\nu^0 \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}, \quad (\nu = 0, 1, 2, \ldots, n)$$

and consequently

$$S_\nu \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}, \quad (\nu = 0, 1, 2, \ldots, n)$$

*Principal domain of the set* $(\Delta)$.

102

Let us apply the general theory introduced in this mémoire to a positive quadratic form

$$f = nx_1^2 + nx_2^2 + \ldots + nx_n^2 - 2x_1x_2 - 2x_1x_3 \ldots - 2x_{n-1}x_n$$

where one has admitted

$$a_{11} = n \text{ and } a_{ij} = -1. \ (i \neq j; i = 1,2,\ldots,n; j = 1,2,\ldots,n) \qquad (1)$$

Let us find all the representations of the minimum of the form $f$ in a set composed of all the systems of integers which are congruous to a system $(l_1, l_2, \ldots, l_n)$ with respect to the modulus 2. Let us admit

$$l_1 = 1, \ i = 1,2,\ldots,\lambda \text{ and } l_i = 0, i = \lambda+1,\ldots,n. \ (\lambda = 1,2,\ldots,n) \qquad (2)$$

The problem described reduces to the study of the minimum of the form

$$f(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n)$$

in the set $E$ composed of all the systems $(x_i)$ of integers $x_1, x_2, \ldots, x_n$.

Let us notice that the form $f$, by virtue of equalities (1), can be written

$$f = \sum_{i=1}^{n} x_i^2 + \sum\sum_{i<j} (x_i - x_j)^2. \qquad (3)$$

Each form

$$x_i^2, \ (i = 1,2,\ldots,n) \ \ (x_i - x_j)^2, \ (i = 1,2,\ldots,n; i < j; j = 1,2,\ldots,n)$$

satisfied, by (2), the condition

$$\left\{ \begin{array}{ll} (l_i + 2x_i)^2 \geq l_i^2, & (i = 1,2,\ldots,n) \\ (l_i - l_j + 2(x_i - x_j))^2 \geq (l_i - l_j)^2, & (i = 1,2,\ldots,n; i < j; j = 1,2,\ldots,n) \end{array} \right. \qquad (4)$$

Whatever may be the integral values of $x_1, x_2, \ldots, x_n$. It follows, by (3), that

$$f(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n) \geq f(l_1, l_2, \ldots, l_n).$$

For the equality

$$\varphi(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n) = \varphi(l_1, l_2, \ldots, l_n)$$

to holds, it is necessary, by (3) and (4), that one had the equalities

$$(l_i + 2x_i)^2 = l_i^2, \ \ l_i - l_j + 2(x_i - x_j) = (l_i - l_j)^2.$$

$$(i = 1,2,\ldots,n; i < j; j = 1,2,\ldots,n)$$

By virtue of (2), one obtains

$$x_i = 0 \text{ or } x_i = -l_i, \quad (i = 1, 2, \ldots, n)$$

therefore the form $f$ possesses only two representations of the minimum $(l_1, l_2, \ldots, l_n)$ and $(-l_1, -l_2, \ldots, -l_n)$ in the set examined.

By attributing to the index $\lambda$ in the inequalities (2) the values $\lambda = 1, 2 \ldots, n$ and by permuting the numbers $l_1, l_2, \ldots, l_n$, one obtains $2^n - 1$ systems which characterise, by virtue of the theorem of Number 48, the faces in $n-1$ dimensions of the parallelohedron $R$ corresponding to the positive quadratic form $f$.

The parallelohedron $R$ will be defined by $2(2^n - 1)$ independent inequalities

$$\begin{aligned}
&1 \cdot n \pm 2x_{k_1} \geq 0, \\
&2 \cdot (n-1) \pm 2(x_{k_1} + x_{k_2}) \geq 0, \quad (k_1 < k_2) \\
&\quad \ldots \\
&\lambda(n - \lambda + 1) \pm 2(x_{k_1} + \ldots + x_{k_\lambda}) \geq 0, \qquad (k_1 < k_2 < \cdots < k_\lambda) \\
&\quad \ldots \\
&n \cdot 1 \pm 2(x_{k_1} + x_{k_2} + \ldots + x_{k_n}) \geq 0, \qquad (k_1 < k_2 < \cdots < k_n)
\end{aligned}$$

where $k_1 = 1, 2, \ldots, n, \ k_2 = 2, \ldots, n, \ldots, \ k_\lambda = \lambda, \ldots, n, \ k_n = n$.

To have more convenience in the subsequent notations, let us write

$$u_0 = x_1 + x_2 \ldots + x_n, \ u_1 = -x_1, \ u_2 = -x_2, \ldots, \ u_n = -x_n \tag{5}$$

and notice that all the sums

$$\pm x_{k_1}, \ \pm(x_{k_1} + x_{k_2}), \ldots, \ \pm(x_{k_1} + x_{k_2} + \ldots + x_{k_n})$$

are expressed by the sums

$$u_{h_0}, \ u_{h_0} + u_{h_1}, \ldots, \ u_{h_0} + u_{h_1} + \ldots + u_{h_{n-1}}$$

where $h_0 < h_1 < h_2 < \cdots < h_{n-1}$ and $h_0 = 0, 1, 2, \ldots, n, \ h_1 = 1, 2, \ldots, n, \ldots, \ h_{n-1} = n-1, \ n$.

The inequalities which define the parallelohedron $R$ can be written

$$\begin{cases}
1 \cdot n + 2u_{h_0} \geq 0, \\
2 \cdot (n-1) + 2(u_{h_0} + u_{h_1}) \geq 0, \ (h_0 < h_1) \\
\quad \ldots \\
n \cdot 1 + 2(u_{h_0} + u_{h_1} \cdots + u_{h_{n-1}}) \geq 0, \ (h_0 < h_1 < \cdots < h_{n-1})
\end{cases} \tag{6}$$

where $h_0 = 0, 1, 2, \dots, n,\ h_1 = 1, 2, \dots, n, \dots,\ h_{n-1} = n-1, n.$

103

Let us find the vertices of the parallelohedron $R$. To this effect, let us examine a point $(\alpha_i)$ verifying the equations

$$n+2u_1 = 0,\ 2(n-1)+2(u_2+u_2) = 0, \dots, n\cdot 1+2(u_1+u_2+\dots+u_n) = 0 \quad (7)$$

By virtue of (5), one obtains

$$\alpha_1 = \frac{1}{2}n,\ \alpha_2 = \frac{1}{2}(n-2),\ \alpha_k = \frac{1}{2}(n-2k+2), \dots, \alpha_n = \frac{1}{2}(-n+2). \quad (8)$$

I argue that the point obtained $(\alpha_i)$ presents a vertice of the parallelohedron $R$. To demonstrate this, let us examine the form

$$f(x_1, x_2, \dots, x_n) + 2\sum_{i=1}^{n} \alpha_i x_i$$

or, by (8), the form

$$f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{n} (n-2i+2)x_i.$$

For the point $(\alpha_i)$ determined by the equalities (8) to be a vertice of the parallelohedron $R$, it is necessary and sufficient that the inequality

$$f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{n} (n-2i+2)x_i \geq 0 \quad (9)$$

holds in the set $E$. By noticing that

$$f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{n} (n-2i+2)x_i =$$
$$\sum_{i=1}^{n} (x_i^2 + x_i) + \sum\sum_{i<j} \left[(x_i - x_j)^2 + x_i - x_j\right],$$

one obtains the inequalities (9) because the inequalities

$$x_i^2 + x_i \geq 0,\ (x_i - x_j)^2 + x_i - x_j \geq 0, \quad (i = 1, 2, \dots, n; j = 2, 3, \dots, n)$$

take place within the set $E$.

For the equality

$$f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{n} (n-2i+2)x_i = 0 \quad (10)$$

to hold, it is necessary and sufficient that one had the equality

$$x_i^2 + x_i = 0,\ (x_i - x_j)^2 + x_i - x_j = 0.(\mathrm{i} = 1, 2, \ldots, \mathrm{n}; \mathrm{i} < \mathrm{j}; \mathrm{j} = 2, 3, \ldots, \mathrm{n})$$

One declares that

$$x_i = -1,\ (i = 1, 2, \ldots, \lambda) \quad x_i = 0.\ (i = \lambda + 1, \ldots, n)$$

By attributing to the index $\lambda$ the values $\lambda = 0, 1, 2 \ldots, n$ one obtains $n + 1$ systems verifying the equality (10).

$$(0, 0, \ldots, 0), (-1, 0, \ldots, 0), (-1, -1, 0, \ldots, 0), \ldots, (-1, -1, \ldots, -1).$$

It is thus demonstrated that the point $(\alpha_i)$ determined by the equations (7) presents a simple vertice of the parallelohedron $R$.

Let us introduce in our studies symbol

$$(h_0, h_1, h_2, \ldots, h_n) \tag{11}$$

in which the indices $h_0, h_1, h_2, \ldots, h_n$ present a permutation of numbers $0, 1, 2, \ldots, n$ and let us agee to indicate by this symbol a point which verifies the equations

$$\begin{gathered} n + 2u_{h_0} = 0,\ 2(n-1) + 2\left(u_{h_0} + u_{h_1}\right) = 0, \ldots, \\ n + 2\left(u_{h_0} + u_{h_1} + \ldots, u_{h_{n-1}}\right) = 0. \end{gathered} \tag{12}$$

By virtue of the definition established of the symbol (11), the vertice $(\alpha_i)$ of the parallelohedron $R$ determined by the equations (7) will be characterised by the symbol

$$(1, 2, \ldots, n, 0).$$

I argue that each symbol (11) characterises a vertex of parallelohedron $R$.

To demonstrate this, let us effect a transformation of the parallelohedron $R$ with the help of a substitution

$$u_1 = u'_{h_0},\ u_2 = u'_{h_1}, \ldots, u_n = u'_{h_{n-1}},\ u_0 = u'_{h_n}, \tag{13}$$

where one has admitted

$$u'_0 = x'_1 + x'_2 + \ldots + x'_n,\ u'_1 = -x'_1, \ldots,\ u'_n = -x'_n.$$

The inequalities (6) which define the parallelohedron $R$ will be permuted by the substitution considered, therefore the parallelohedron $R$ will be transformed into itself.

To vertice $(\alpha_i)$ of the parallelohedron $R$ determined by the equations (7) will be transformed, by virtue of (13), into a vertice of the parallelohedron $R$ determined by the equations (12), therefore the vertice will be characterised by the symbol (11).

We have demonstrated the existence of $(n+1)!$ simple vertices of the parallelohedron $R$ corresponding to the positive quadratic form $\varphi$. As the number of vertices of any one parallelohedron corresponding to a positive quadratic form does not exceed a limit $(n+1)!$, by virtue of the formula (3) of Number 101, one concludes that the parallelohedron $R$ does not possess vertices other than those which are characterised by the symbol

$$(h_0, h_1, \dots, h_n)$$

in which one permutes the indices $0, 1, 2, \dots, n$ in every possible ways.

All the vertices of the parallelohedron $R$ are simple, therefore the parallelohedron $R$ is primitive. By noticing that the number of vertices of the parallelohedra $R$ is expressed by the formula

$$S_0 = (n+1)! = (n+1)\Delta^{(m)}(m^n)_{m=1}, \tag{14}$$

one concludes, by virtue of that which has been said in Number 66, that the number $S_\nu$ of faces in $\nu$ dimensions of the parallelohedron $R$ is expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \dots, n)$$

104

Let us find the regulators and the characteristics of faces in $n-1$ dimensions of simplexes of the set $(L)$ which defines the type of primitive parallelohedra to which belongs the parallelohedron $R$ examined.

Any symbol $(h_0, h_1, \dots, h_n)$ defines a simplex characterised by the linear functions

$$u_{h_0}, u_{h_0} + u_{h_1}, \dots, u_{h_0} + u_{h_1} + \dots + u_{h_n}.$$

By virtue of (5), one will have identically

$$u_{h_0} + u_{h_1} + \dots + u_{h_n} = 0. \tag{15}$$

Notice that $n+1$ simplexes which one obtains by carrying out the circular permutations of indices $h_0, h_1, \dots, h_n$

$$(h_0, h_1, \dots, h_n), (h_1, h_2, \dots, h_0), \dots, (h_n, h_0, \dots, h_{n-1})$$

are congruent. By choosing a representative among these simplexes, one will determine in this manner $n!$ incongruent simplexes of the set $(L)$.

Let us examine two simplexes determined by two symbols

$$(h_0, h_1, h_2, \ldots, h_n) \quad \text{and} \quad (h_1, h_0, h_2, \ldots, h_n)$$

which differ only by a transposition of indices $h_0$ and $h_1$.

By virtue of the definition established, these simplexes are characterised by the functions

$$[u_{h_0}, u_{h_0} + u_{h_1}, u_{h_0} + u_{h_1} + u_{h_2}, \ldots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}] \tag{16}$$

and

$$[u_{h_1}, u_{h_1} + u_{h_0}, u_{h_1} + u_{h_0} + u_{h_2}, \ldots, u_{h_1} + u_{h_0} + \ldots + u_{h_n}]\,. \tag{17}$$

These two simplexes differ only by the vertices which are characterised by the function $u_{h_0}$ and $u_{h_1}$.

One concludes that these two simplexes are contiguous by a face in $n-1$ dimensions which is characterised by the functions

$$[u_{h_0} + u_{h_1}, u_{h_0} + u_{h_1} + u_{h_2}, \ldots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}]\,. \qquad ([1]8)$$

Let us determine the characteristic $\pm(p_i)$ of this face. By declaring, as that which we have done in Number 72,

$$u^0_{h_0} + u^0_{h_1} = \delta, u^0_{h_0} + u^0_{h_1} + u^0_{h_2} = \delta, \ldots, u^0_{h_0} + \ldots, + u^0_{h_1} + u^0_{h_n} = \delta,$$

one obtains, by (15), $\delta = 0$ and consequently

$$u^0_{h_0} = -u^0_{h_1}, u^0_{h_2} = 0, \ldots, u^0_{h_n} = 0. \tag{19}$$

By indicating with $(p_i)$ the characteristic of the face (18) with regard to the simplex (16), one will have a supplementary condition

$$u^0_{h_0} > 0 \tag{20}$$

which, added to the equalities (19), well defines the characteristic $(p_i)$.

Let us indicate, to make short,

$$h_0 = i \quad \text{and} \quad h_1 = j \tag{21}$$

and suppose that $i \neq 0$ and $j \neq 0$. By virtue of equalities (5), one will have

$$u^0_{h_0} = u^0_i = -p_i, u^0_{h_1} = u^0_j = -p_j.$$

By virtue of (19) and (20), one obtains

$$\begin{cases} p_k = 0, \\ p_i = -1, p_j = 1. \end{cases} \quad (k = 1, 2, \ldots, n; k \neq i; k \neq j) \tag{22}$$

One can therefore characterise the characteristic $(p_i)$ by a corresponding function

$$\sum p_i x_i = -x_i + x_j.$$

Let us suppose that $j = 0$. One will have in this case the equalities

$$p_1 + p_2 + \ldots + p_n = p_i, p_k = 0 \quad (k = 1, 2, \ldots, n; k \neq i)$$

and consequently, by (20),

$$\sum p_i x_i = -x_i.$$

In the same way one will examine the case $i = 0$. One can bring together the three cases examined by indicating the characteristic of the face (18) by the function $-x_i + x_j$, provided that $x_0 = 0$.

Let us find the regulator $\rho_{ij}$ of the face examined. To that effect, let us determine the number $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ after the conditions

$$u_{h_1} = \sum_{\lambda=0}^{n} \vartheta_\lambda (u_{h_0} + \ldots + u_{h_\lambda}) \text{ where } \sum \vartheta_\lambda = 1.$$

One obtains

$$\vartheta_0 = -1, \vartheta_1 = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = 1.$$

By applying the formula (20) of Number 72, one finds

$$2\rho_{ij} \left[ (u^0_{h_j} + \ldots + u^0_{h_\lambda}) - u^0_{h_1} \right] = (u_{h_1})^2 + (u_{h_0})^2 - (u_{h_0} + u_{h_1})^2$$
$$- (u_{h_0} + u_{h_1} + \ldots + u_{h_n})^2 \text{ where } \lambda > 0.$$

By virtue of equalities (15) and (19), this formula comes down to the one here

$$2\rho_{ij} = (u_{h_1})^2 + (u_{h_0})^2 - (u_{h_0} + u_{h_1})^2,$$

and consequently

$$\rho_{ij} = -u_{h_0} u_{h_1}$$

or, by (21)

$$\rho_i = -u_i u_j. \tag{23}$$

Let us suppose that $j = 0$; the formulae (5) give

$$\rho_{i0} = x_i(x_1 + x_2 + \ldots + x_n) = x_1 x_i + x_2 x_i + \ldots + x_n x_i.$$

By replacing in this formula $x_ix_j$ by $a_{ij}$, one obtains the sought-for expression of the regulator $\rho_{i0}$

$$\rho_{i0} = \sum_{k=1}^{n} a_{ki} \quad (i = 1, 2, \ldots, n) \tag{24}$$

By supposing that $i \neq 0$ and $j \neq 0$, one will have

$$\rho_{ij} = -x_ix_j$$

and consequently

$$\rho_{ij} = -a_{ij} \quad (i = 1, 2, \ldots, \nu; i \neq j; j = 1, 2, \ldots, n) \tag{25}$$

Observe that the face (18) possesses the regulator $\rho_{ij}$ and the characteristic $-x_i + x_j$ in addition to values of indices $h_2, \ldots, h_n$. One concludes that there exist $(n-1)!$ inconcruent faces of simplexes of the set $(L)$ which possess the same regulator $\rho_{ij}$ and the same characteristic

$$-x_0 + x_j. \quad (i = 0, 1, 2, \ldots, n; i \neq j; j = 0, 1, 2, \ldots, n)$$

By applying the formula (I.) from Number 84, one obtains

$$\sum\sum a_{ij}x_ix_j = \sum_{i<j}\sum \rho_{ij}(x_i - x_j)^2,$$

$$(i = 0, 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

where one has admitted $x_0 = 0$, or differently

$$\sum\sum a_{ij}x_ix_j = \sum_{i=1}^{n} \rho_{i0}x_i^2 + \sum_{i<j}\sum \rho_{ij}(x_i - x_j)^2. \tag{26}$$

$$(i = 1, 2, \ldots, n; i < j; j = 2, 3, \ldots, n)$$

The domain $\Delta$ of quadratic forms corresponding to the type of primitive parallelohedra examined will be determined by the inequalities

$$\rho_{ij} \geq 0 \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n)$$

or differently, according to (24) and (25), by the inequalities

$$\sum_{k=1}^{n} a_{ki} \geq 0, \quad -a_{ij} \geq 0. \quad (i = 1, 2, \ldots, n; i < j; j = 2, 3, \ldots, n) \tag{27}$$

The number of these inequalities is equal to $\frac{n(n+1)}{2}$, thus the domain of quadratic forms defined by these inequalities is a simple domain.

By attributing to the parameters $\rho_{ij}$ $(i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n; i < j)$ in the formula (26) the positive arbitrary values or zeros, one will determine all the quadratic forms belonging to the domain $\Delta$.

One remarkable coincidence is signalling. The domain of quadratic forms (27) has been studied in my first mémoire cited † where it has been called principal domain. This domain corresponds to a principal perfect positive quadratic form

$$\varphi = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_2 + \ldots + x_{n-1}x_n.$$

It is remarkable that the set of characteristics found

$$\pm x_i,\ \pm(x_i - x_j),\ (i = 1, 2, \ldots, n; i < j; j = 2, 3, \ldots, n)$$

coincides with the set of representations of the minimum of the principal perfect form $\varphi$.

*Domains of quadratic forms contiguous to the principal domain.*

105 All the faces in $\frac{n(n+1)}{2} - 1$ dimensions of the principal domain $\Delta$ are equivalent. †

It follows that all the domains of forms belonging to the set $(\Delta)$ which are contiguous to the principal domain $\Delta$ by the faces in $\frac{n(n+1)}{2} - 1$ dimensions are equivalent.

In the case $n = 2$ and $n = 3$, the set $(\Delta)$ of domains of quadratic forms is made up of a single class of domains equivalent to the principal domain.

One concludes that in the space of 2 and of 3 dimensions there is only a single type of primitive parallelohedra, provided that one does not consider as different the equivalent types which correspond to the equivalent domains of quadratic forms.

Let us suppose that $n \geq 4$ and find the domain $\Delta'$ which is contiguous to the principal domain $\Delta$ by the face determined with the help of the equation

$$\rho = -a_{12} = 0.$$

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† This Journal, V. 133

† See my mémoire cited

By applying the algorithm explained in Number 96, let us determine the incongruent convex polyhedra which correspond to the independent regulator $\rho$.

We have seen in Number 104 that the regulator $\rho = \rho_{12}$ corresponds to the common faces of simplexes defined by the symbols

$$(1, 2, h_1, \ldots, h_n), (2, 1, h_2, \ldots, h_n)$$

where $h_2, h_3, \ldots, h_n$ present an arbitrary permutation of indices $0, 3, 4, \ldots, n$.

The two corresponding simplexes are characterised by the functions

$$[u_1, u_1 + u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_n}]$$

and

$$[u_2, u_2 + u_1, u_2 + u_1 + u_{h_2}, \ldots, u_2 + u_1 + \ldots + u_{h_n}] .$$

By declaring

$$u_2 = \vartheta_0 u_1 + \vartheta_1(u_1 + u_2) + \vartheta_2(u_1 + u_2 + u_{h_2}) + \ldots + \vartheta_n(u_1 + u_2 + \ldots + u_{h_n})$$

where $\sum_{k=0}^{n} = 1$, one obtains

$$\vartheta_0 = -1, \vartheta_1' = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = 1. \qquad (3)$$

As among the numbers obtained is found only one negative number $\vartheta_0$, one concludes that the two simplexes (1) and (2) make up a polyhedron $K$ corresponding to the independent regulator $\rho$.

Let us indicate by $(L')$ the set of simplexes which characterise the domain $\Delta'$ of quadratic forms. By virtue of that which has been said in Number 91, the polyhedron $K$ in the set $(L')$ will be made up from simplexes which one obtains by replacing the vertices of the simplex (1) which correspond to the positive values of numbers (3) by the vertice characterised by the function $u_2$. As in the series (3) only two positive numbers $\vartheta_1$ and $\vartheta_n$ are found, one obtains two simplexes characterised by the functions

$$[u_1, u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_n}] \qquad (4)$$

and

$$[u_1, u_1 + u_2, \ldots, u_1 + u_2 + \ldots + u_{h_{n-1}}, u_2] . \qquad (5)$$

These two simplexes make up the polyhedron $K$ and replace the two simplexes (1) and (2) in the set $(L')$.

By effecting all the permutation of indices $h_2, \dots, h_n$, one obtains $(n-1)!$ incongruent convex polyhedra which correspond to the independent regulator $\rho$.

By replacing in the set $(L)$ the simplexes congruent to the simplexes (1) and (2) by the simplexes which are congruent to the simplexes (4) and (5), one will reconstruct the set $(L)$ of simplexes into a set $(L')$ which characterises the domain $\Delta'$.

Notice that the number of incongruent simplexes of the set $(L')$ is equal to $n!$ also. It follows that the number of faces in $\nu$ dimensions of primitive parallelohedra belonging to the type characterised by the domain of form $\Delta'$ is expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \dots, n)$$

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Let us find the regulators and the characteristic of faces in $n-1$ dimensions of simplexes belonging to the set $(L')$.

Let us examine in the first place the simplexes contiguous to a face in $n-1$ dimensions which belong to the set $(L)$ and to the set $(L')$.

The condition necessary and sufficient for which the two simplexes characterised by the symbols

$$(h_0, h_1, h_2, \dots, h_n) \text{ and } (h_1, h_0, h_2, \dots, h_n), \tag{6}$$

which are contiguous by a face in $n-1$ dimensions, also belong to the set $(L')$, consists in so long as within the two series

$$h_0, h_1, h_2, \dots, h_n, h_0 \quad \text{and} \quad h_1, h_0, h_2, \dots, h_n, h_1$$

the indices 1 and 2 are not adjacent. By declaring

$$h_0 = i \quad \text{and} \quad h_1 = j,$$

one obtains $(n-1)! - 2(n-2)!$ pairs of symbols (6) which satisfy the condition assumed.

By indicating with $\rho_{ij}$ the regulator and with $\pm(x_i - x_j)$ the characteristic of the face common to the simplexes (6) determined in the set $(L)$, one will have for the set $(L')$ the same regulator

$$\rho'_{ij} = \rho_{ij}, \quad (i = 0, 1, 2, \dots, n; i < j; j = 1, 2, \dots, n)$$

and the same characteristic $\pm(x_i - x_j)$, the regulator $\rho_{12}$ being excluded. The regulator obtained $\rho'_{ij}$ and the characteristic $\pm(x_i - x_j)$ belong to $(n-1)! - 2(n-2)!$ incongruent faces in $n-1$ dimensions of the set $(L')$ of simplexes.

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This declared, let us examine the regulators and the characteristics of faces of simplexes (4) and (5) which make up the polyhedron $K$ in the set $(L')$.

The first group of faces of the simplex (4) is composed of two faces which are opposite to the vertices $u_1$ and $u_2$. The first face is characterised by the functions

$$u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1, u_2, \ldots, u_{h_n}. \tag{7}$$

This face belongs in the set $(L)$ to the simplexes characterised by the symbols

$$(2, 1, h_2, h_3, \ldots, h_n) \quad \text{and} \quad (2, h_2, 1, h_3, \ldots, h_n).$$

The second simplex also belongs to the set $(L')$. It follows that the simplex (4) is contiguous to the simplex $(2, h_2, 1, h_3, \ldots, h_n)$ by the face (7).

Let us declare $h_2 = i$ where $i = 0, 3, \ldots, n$ and indicate by $\rho'_{i1}$ the regulator and by $\pm(x_1 - x_i)$ the characteristic of the face (7) in the set $(L')$. By applying the formula (8) of Number 93, one obtains

$$\rho'_{i1} = \rho_{i1} + \rho. \ \ (i = 0, 3, \ldots, n)$$

In the same manner, one will examine the regulators of the face of the simplex (4) which is opposite to the vertex $u_2$. By putting $h_2 = i$, one will have

$$\rho'_{i2} = \rho_{i2} + \rho. \ \ (i = 0, 3, \ldots, n) \tag{9}$$

Examine the first group of faces of the simplex (5). This group is made up of two faces which are opposite to the vertices $u_1$ and $u_2$. The first face is characterised by the functions.

$$u_1 + u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_{n-1}}, u_2.$$

This face belongs in the set $(L)$ to the simplex $(2, 1, h_2, \ldots, h_n)$ and to the simplex congruent to the simplex $(h_n, 1, h_2, \ldots, h_{n-1}, 2)$ and which is characterised by the functions

$$u_2, u_2 + u_1, u_2 + u_1 + u_{h_2}, \ldots, u_2 + u_1 + \ldots + u_{h_{n-1}}, u_2 - u_{h_n}.$$

This simplex also belongs to the set $(L')$. By putting $h_n = i$ and by applying the formula (8) of Number (93), one finds

$$\rho'_{i2} = \rho_{i2} + \rho$$

The characteristic of this face will be $\pm(x_2 - x_i)$ $(i = 0, 3, \ldots, n)$.

In the same way one will examine the face of the simplex (5) which is opposite to the vertex $u_2$.

One will obtain by letting $h_n = i$

$$\rho'_{i1} = \rho_{i1} + \rho$$

and the characteristic will be $\pm(x_1 - x_i)$, $(i = 0, 3, \ldots, n)$.

Let us notice that the number of incongruent faces, belonging to the first group of simplexes of the set $(L')$, which possess the regulator determined by the formula (8) or by the formula (9), is equal to $2(n-2)!$.

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The second group of faces in $n-1$ dimensions of the simplex (4) is composed of $n-2$ faces which are opposite to the vertices

$$u_1 + u_2 + u_{h_2}, u_1 + u_2 + u_{h_2} + u_{h_3}, \ldots, u_1 + u_2 + \ldots, u_{h_{n-1}}.$$

Let us examine a face which is opposite to the vertice $u_1 + u_2 + \ldots + u_{h_k}$ $(k = 2, 3, \ldots, n-1)$.

A transposition of indices $h_k$ and $h_{k+1}$ in the symbol $(*)$ leads to the symbol

$$[u_1, u_2, \ldots, u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}}, u_1 + u_2 + \ldots + u_{h_{k+1}} + u_{h_k}, \\ \ldots, u_1 + u_2 + \ldots + u_{h_n}]$$

which defines a simplex belonging to the set $(L')$ and which is contiguous to the simplex (4) by the face in $n-1$ dimensions examined.

Let us write $h_k = i, h_{k+1} = j$ and indicate by $\rho'_{ij}$ the corresponding regulator. The corresponding characteristic will be determined by the equations

$$u_1^0 = 0, u_2^0 = 0, \ldots, u_1^0 + u_2^0 + \ldots + u_{h_{k-1}}^0 = 0,$$

$$u_1^0 + u_2^0 + \ldots + u_{h_{k+1}}^0 = 0, u_1^0 + u_2^0 + \ldots + u_{h_n}^0 = 0.$$

One obtains

$$u_i^0 = -u_j^0 \text{ and } u_r^0 = 0. \quad (r = 0, 1, 2, \ldots, n; r \neq i; r \neq j)$$

It follows that the characteristic will be represented by the function $\pm(x_i - x_j)$.

By declaring

$$u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}} = \vartheta_0 u_1 + \vartheta_1 u_2 + \vartheta_2 (u_1 + u_2 + u_{h_2}) + \ldots \\ + \vartheta_k (u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_k}) + \ldots + \vartheta_n (u_1 + u_2 + \ldots + u_{h_n}) \tag{10}$$

where $\sum_{r=0}^{n} \vartheta_r = 1$, one obtains

$$\vartheta_0 = 0, \vartheta_1 = 0, \ldots, \vartheta_{k-2} = 0, \vartheta_{k-1} = 1, \vartheta_k = -1, \vartheta_{k+1} = 1, \vartheta_{k+2} = 0, \ldots, \vartheta_n = 0,$$

provided that $k > 2$.

The regulator $\rho'_{ij}$ will be determined by the formula

$$\begin{aligned} 2\rho'_{ij} =& (u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}})^2 - (u_1 + u_2 + \ldots + u_{h_{k-1}})^2 + \\ & (u_1 + u_2 + \ldots + u_{h_k})^2 - (u_1 + u_2 + \ldots + u_{h_{k+1}})^2. \end{aligned}$$

After the reductions, one finds

$$\rho'_{ij} = -u_{h_k} u_{h_{k+1}}$$

or differently

$$\rho'_{ij} = -u_i u_j,$$

thus, by virtue of the formula (23) of Number 104, one will have

$$\rho'_{ij} = \rho_{ij}. \ (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n; k = 3, 4, \ldots, n-1) \qquad (11)$$

Let us examine the case $k = 2$. The equality (10) gives in this case

$$\vartheta_0 = 1, \vartheta_1 = 1, \vartheta_2 = -1, \vartheta_3 = 1, \vartheta_4 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = -1,$$

and consequently

$$\begin{aligned} 2\rho'_{ij} =& (u_1 + u_2 + u_{h_3})^2 - u_1^2 - u_2^2 + (u_1 + u_2 + u_{h_2})^2 \\ & - (u_1 + u_2 + u_{h_2} + u_{h_3})^2 + (u_1 + u_2 + \ldots + u_{h_n})^2. \end{aligned}$$

As $u_{h_2} = i$, $u_{h_3} = j$ and $u_1 + u_2 + \ldots + u_{h_n} = 0$, one obtains

$$\rho'_{ij} = u_1 u_2 - u_i u_j,$$

thus one will have in the case examined

$$\rho'_{ij} = \rho_{ij} - \rho. \ (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n) \qquad (12)$$

In the same way, one will examine the faces of the simplex (5) belonging to the second group and one will obtain the same formulae (11) and (12).

Let us notice that the number of incongruent faces of simplexes of the set $(L')$ which belong to the second group and the regulator of which is determined by the formula (11) is equal to $(n-3)!2(n-3)$. The number of regulators which are determined by the formula (12) is equal to $2(n-3)!$.

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The third group of faces of simplexes (4) and (5) is composed of a single face

$$\left[u_1, u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_{n-1}}\right]$$

which is common to these two simplexes.

The characteristic of this face is determined by the equations

$$u_1^0 = \delta, u_2^0 = \delta, u_1^0 + u_2^0 + u_{h_2}^0 = \delta, \ldots, u_1^0 + u_2^0 + \ldots + u_{h_{n-1}}^0 = \delta.$$

It results in that

$$u_1^0 = u_2^0 = \delta, u_2^0 = -\delta, u_{h_3}^0 = 0, \ldots, u_{h_{n-1}}^0 = 0, u_{h_n}^0 = \delta.$$

One concludes that $\delta = \pm 1$. By admitting

$$h_2 = i \quad \text{and} \quad h_n = j,$$

one obtains the characteristic $\pm(x_1 + x_2 - x_i - x_j)$.

Let us indicate the corresponding regulator by $\rho'_{ij}$. with the help of equalities

$$\begin{aligned} u + u_2 =& \vartheta_0 u_1 + \vartheta_1 u_2 + \vartheta_2 (u_1 + u_2 + u_{h_2}) + \ldots \\ & + \vartheta_n (u_1 + u_2 + \ldots + u_{h_n}) \text{ where } \sum_{k=1}^{n} \vartheta_k = 1, \end{aligned}$$

one obtains

$$\vartheta_0 = 1, \vartheta_1 = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = -1.$$

The regulator $\rho'_{ij}$ will be determined by the formula

$$2\rho'_{ij} = (u_1 + u_2)^2 - u_1^2 - u_2^2 - (u_1 + u_2 + \ldots + u_{h_n}),$$

and it becomes

$$\rho'_{ij} = u_1 u_2,$$

thus

$$\rho'_{ij} = -\rho. \quad (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n)$$

The number of incongruent faces belonging to the third group having the characteristic $\pm(x_1 + x_2 - x_i - x_j)$ is equal to $2(n-3)!$.

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With the help of deduced formulae, one can determine all the independent regulators. Let us admit

$$\rho'_{12} = -\rho, \ \rho'_{i1} = \rho i1 + \rho, \ \rho'_{i2} = \rho i2 + \rho, \ \rho'_{ij} = \rho_{ij}. \tag{13}$$

$$(i = 0, 3, \dots, n; i < j; j = 3, \dots, n)$$

One obtains in this manner $\frac{n(n+1)}{2}$ independent regulators $\rho'_{ij}$ $(i = 0, 1, 2, \dots, n; i < j; j = 1, 2, \dots, n)$.

The results of our studies can be gathered in the following table:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1. Regulator | $\rho'_{12}$, | characteristic | $\pm(x_1+x_2-x_i-x_j)$, | their number | $2(n-3)!$ |
| 2. Regulator | $\rho'_{i1}$, | characteristic | $\pm(x_1-x_i)$, | their number | $2(n-2)!$ |
| 3. Regulator | $\rho'_{i2}$, | characteristic | $\pm(x_2-x_i)$, | their number | $2(n-2)!$ |
| 4. Regulator | $\rho'_{i1}+\rho'_{12}$, | characteristic | $\pm(x_1-x_i)$, | their number | $(n-1)!-2(n-2)!$ |
| 5. Regulator | $\rho'_{i2}+\rho'_{12}$, | characteristic | $\pm(x_2-x_i)$, | their number | $(n-1)!-2(n-2)!$ |
| 6. Regulator | $\rho'_{ij}$, | characteristic | $\pm(x_i-x_j)$, | their number | $(n-1)!-2(n-3)!$ |
| 7. Regulator | $\rho'_{ij}+\rho'_{12}$, | characteristic | $\pm(x_i-x_j)$, | their number | $2(n-3)!$ |

The indices $i$ and $j$ are the values $0, 3, \dots, n$ and one has admitted $x_0 = 0$.

The domain $\Delta'$ of quadratic forms corresponding to the set $(L')$ of simplexes is determined by $\frac{n(n+1)}{2}$ independent inequalities

$$\rho'_{ij} \geq 0. \quad (i = 0, 1, 2, \dots, n; j = 0, 1, 2, \dots, n; i \neq j)$$

As a result, the domain $\Delta'$ is simple.

By applying the formula (I) of Number 84, one will determine, by (13), any quadratic form $\sum\sum a_{ij}x_ix_j$ by the following formula

$$\sum\sum a_{ij}x_ix_j = \sum_{i<j}\sum \rho'_{ij}(x_i - x_j)^2 + \rho'_{12}\omega, \tag{14}$$

$$(i = 0, 1, 2, \dots, n; j = 1, 2, \dots, n)$$

where one has admitted $x_0 = 0$ and

$$\omega = (n-2)x_1^2 + (n-2)x_2^2 + 2x_3^2 + \dots + 2x_n^2 + 2x_1x_2 + 2x_1x_3 - \dots \\ - 2x_1x_n - 2x_2x_3 - \dots - 2x_2x_n.$$

By attributing to the independent parameters $\rho'_{ij}$ $(i = 0, 1, 2, \dots, n;$ $j = 0, 1, 2, \dots, n)$ all the positive values or null, one will determine by the formula (14) all the quadratic forms belonging to the domain $\Delta'$.

One coincidence is to be pointed out: the domain $\Delta'$ presents a part of the domain $R$, corresponding to the perfect form $\varphi$, which has been determined in my mémoire cited. The set composed of linear forms

$$\pm(x_i - x_j) \quad (i = 0, 1, 2, \dots, n; j = 0, 1, 2, \dots, n)$$

(the form $\pm(x_1 - x_2)$ being excluded) and of forms

$$(x_1 + x_2 - x_i - x_j) \quad (i = 0, 3, \dots, n; j = 0, 3, \dots, n)$$

where one has put $x_0 = 0$, coincides with the linear forms which define all the representations of the minimum of the perfect form $\varphi_1$.

*Parallelohedra in two dimensions*

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The set $(\Delta)$ of domains of binary quadratic forms is composed of a single class of domains which are equivalent to the principal domain $\Delta$ determined by the inequalities

$$a + b \geq 0, \quad -b \geq 0, \quad c + b \geq 0.$$

Here are the conditions of reduction of binary positive quadratic forms $ax^2 + 2bxy + cy^2$ due to Selling. ‡ Any quadratic form belonging to the principal domain $\Delta$ can be determined by the equalities

$$ax^2 + 2bxy + cy^2 = \lambda x^2 + \mu y^2 + \nu(x - y)^2 \text{ where } \lambda > 0, \ \mu \geq 0, \ \nu \geq 0.$$

The parameters $\lambda, \mu$ and $\nu$ present the regulators of the hexagon of Lejeune Dirichlet ‡ defined by the inequalities

$$\begin{aligned} -\frac{1}{2}(\lambda + \nu) \leq x \leq \frac{1}{2}(\lambda + \nu), \\ -\frac{1}{2}(\mu + \nu) \leq y \leq \frac{1}{2}(\mu + \nu), \\ -\frac{1}{2}(\lambda + \mu) \leq x + y \leq \frac{1}{2}(\lambda + \mu), \end{aligned} \quad .$$

By attributing to the arbitrary parameters $\lambda, \mu, \nu$ the positive values, one will determine by these inequalities a primitive parallelohedron in two dimensions, that is to say a hexagon of Lejeune Dirichlet.

By nullifying one of the parameters $\lambda, \mu, \nu$, for example $\nu$, one will obtain four independent inequalities.

$$\begin{aligned} -\frac{1}{2}\lambda \leq x \leq \frac{1}{2}\lambda, \\ -\frac{1}{2}\mu \leq y \leq \frac{1}{2}\mu, \end{aligned}$$

which define a nonprimitive parallelohedron in two dimensions, it is a parallelogram.

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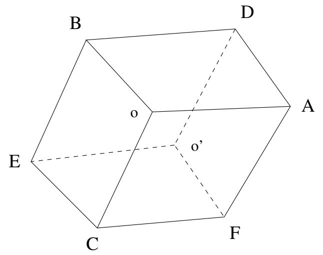
‡ See the Introduction

It is easy to demonstrate that other nonprimitives of the space in two dimensions do not exist.

Each hexagon of Lejeune Dirichlet can be constructed from three parallelograms as is indicated in Fig. 1.

Fig. 1



One of the three parallelograms $OADB$, $OBEC$ and $OCFA$ which form the hexagon $ADBECF$ can be arbitrarily chosen. By choosing, for example, an arbitrary parallelogram $OADB$, one will determine the two remaining parellograms $OBEC$ and $OCFA$ by taking an arbitrary vector $OC$, provided that by extending this vector in the inverse direction one passes through the chosen parallelogram $OADB$.

Observe that in general the point $O$ does not present the centre of the hexagon $ADBECF$.

One can make up the same hexagon of three parallelograms $O'DBE$, $O'ECF$ and $O'FAD$. One concludes that the hexagon of Lejeune Dirichlet does not present anything other than projection of a parallelepiped on the plane.

*Parallelohedra in three dimensions*

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The set $(\Delta)$ of domains of ternary quadratic forms is composed of a single class of domains equivalent to the principal domain $\Delta$ determined by the inequalities

$$a+b'+b'' \geq 0, a'+b''+b \geq 0, a''+b+b' \geq 0, -b \geq 0, -b' \geq 0, -b'' \geq 0.$$

Here are the conditions of reduction of ternary positive quadratic forms $ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy$ due to Selling.

Any ternary quadratic form belonging to the principal domain $\Delta$ can be determined by the equalities

$$\begin{aligned} &ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = \\ &\qquad \lambda x^2 + \lambda' y^2 + \lambda'' z^2 + \mu(y-z)^2 + \mu'(z-x)^2 + \mu''(x-y)^2. \end{aligned}$$

All the primitive parallelohedra in three dimensions can be transformed with the help of linear substitutions into primitive parallelohedra determined by 14 independent inequalities

$$\left\{\begin{array}{lrcll} (1') & -\tfrac{1}{2}(\lambda + \mu' + \mu'') & \leq x \leq & \tfrac{1}{2}(\lambda + \mu' + \mu''), & (1'), \\ (2') & -\tfrac{1}{2}(\lambda' + \mu'' + \mu) & \leq y \leq & \tfrac{1}{2}(\lambda' + \mu'' + \mu), & (2'), \\ (3') & -\tfrac{1}{2}(\lambda'' + \mu + \mu') & \leq z \leq & \tfrac{1}{2}(\lambda'' + \mu + \mu'), & (3'), \\ (4') & -\tfrac{1}{2}(\lambda' + \lambda'' + \mu' + \mu'') & \leq y + z \leq & \tfrac{1}{2}(\lambda' + \lambda'' + \mu' + \mu''), & (4'), \\ (5') & -\tfrac{1}{2}(\lambda_{\prime\prime} + \lambda + \mu_{\prime\prime} + \mu) & \leq z + x \leq & \tfrac{1}{2}(\lambda_{\prime\prime} + \lambda + \mu_{\prime\prime} + \mu), & (5'), \\ (6') & -\tfrac{1}{2}(\lambda + \lambda_{\prime} + \mu + \mu') & \leq x + y \leq & \tfrac{1}{2}(\lambda + \lambda_{\prime} + \mu + \mu'), & (6'), \\ (7') & -\tfrac{1}{2}(\lambda + \lambda' + \lambda'') & \leq x + y + z \leq & \tfrac{1}{2}(\lambda + \lambda' + \lambda''), & (7') \end{array}\right. \tag{2}$$

The parameter $\lambda, \lambda_{\prime}, \lambda_{\prime\prime}, \mu, \mu_{\prime}, \mu_{\prime\prime}$ present the independent regulators of the primitive parallelohedron defined by these inequalities and corresponding to a ternary positive quadratic form (1).

By virtue of the formula (14) of Number 103, any primitive parallelohedron of the space in three dimensions possesses 24 vertices which can be characterised by three numbers corresponding to the different faces in two dimensions of the parallelohedron defined by the inequalities (2).

One will divide these (24) vertices into three groups I, II and III:

| | | | | |
|---|---|---|---|---|
| I | 1 6 7 | $1'$ 2 4 | $2'$ $6'$ 3 | $4'$ $3'$ $7'$ |
| | $1'$ $6'$ $7'$ | 1 $2'$ $4'$ | 2 6 $3'$ | 4 3 7 |
| II | 1 7 5 | $1'$ 4 3 | $4'$ $7'$ $2'$ | $3'$ 2 $5'$ |
| | $1'$ $7'$ $5'$ | 1 $4'$ $3'$ | 4 7 2 | 3 $2'$ 5 |
| III | 1 $3'$ 6 | $1'$ $5'$ 2 | 5 3 7 | $2'$ $7'$ $6'$ |
| | $1'$ 3 $6'$ | 1 5 $2'$ | $5'$ $3'$ $7'$ | 2 7 6 |

Each line of this table is composed of four congruent vertices. In each group the second line is formed from vertices opposite to those which are found in the first line.

Let us examine the regulators and the characteristics of edges of the primitive parallelohedron in three dimensions. It suffices to examine the regulators and the characteristics of faces in two dimensions belonging to three simplexes

$$(0, 1, 6, 7), \quad (0, 1, 7, 5), \quad (0, 1, 3', 6).$$

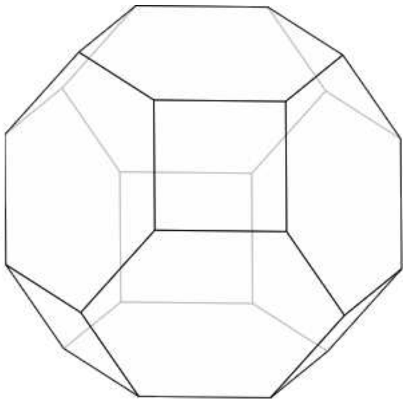
The results of these studies can be brought together in the following table:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| I | 0 1 6 7 | $-1\ 0\ \ 0$<br>$2\ 1\ \ 1$ | $\lambda$ | $1\ -1\ \ 0$<br>$0\ \ 1\ \ 0$ | $\mu''$ | $0\ \ 1\ -1$<br>$1\ \ 0\ \ 1$ | $\mu$ | $0\ \ 0\ \ 1$<br>$0\ \ 0\ -1$ | $\lambda''$ |
| II | 0 1 7 5 | $-1\ 0\ \ 0$<br>$2\ 1\ \ 1$ | $\lambda$ | $1\ \ 0\ -1$<br>$0\ \ 0\ \ 1$ | $\mu'$ | $0\ \ 1\ \ 0$<br>$0\ -1\ \ 0$ | $\lambda'$ | $0\ -1\ \ 1$<br>$1\ \ 1\ \ 0$ | $\mu$ |
| III | 0 1 $3'$ 6 | $-1\ 0\ \ 1$<br>$1\ 0\ -1$ | $\mu'$ | $1\ -1\ \ 0$<br>$0\ \ 1\ \ 9$ | $\mu''$ | $0\ \ 0\ -1$<br>$1\ \ 1\ \ 1$ | $\lambda_{\prime\prime}$ | $0\ \ 1\ \ 0$<br>$0\ -1\ -1$ | $\lambda_{\prime}$ |

The first line contains the characteristics and the regulators of different faces of the corresponding simplexes. The second line is composed of vertices which define the simplexes contiguous to the corresponding simplex by the faces the characteristics of which are found above, in the first line.

The faces of primitive parallelohedron $R$ in three dimensions (Fig. 2) are divided into 8 hexagons of Lejeune Dirichlet and into 6 parallelograms.

Fig. 2



The hexagonal faces of the primitive parallelohedron $R$ are characterised by the numbers

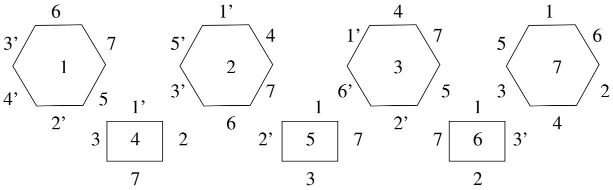
$$1, 2, 3, 7, 1', 2', 3', 7'.$$

The parallelogrammatic faces are characterised by the numbers

$$4, 5, 6, 4', 5', 6'.$$

The faces, the edges and the vertices of the primitive parallelohedron are systematically characterised in Fig. 3.

Fig. 3



One has indicated in this figure the numbers of faces which are contiguous to one of 7 incongruent faces. Each edge is characterised by two adjacent numbers and each vertex by three numbers.

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By nullifying one or more parameters $\lambda, \lambda', \lambda'', \mu, \mu', \mu''$ in the inequalities (2), one will determine the nonprimitive parallelohedra in three dimensions. It is easy to see that the nonprimitive parallelohedra obtained are divided into four different spaces. (Fig. 4–7)

Fig. 4

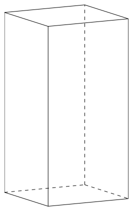


Fig. 5

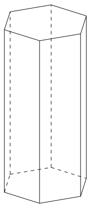


Fig. 6

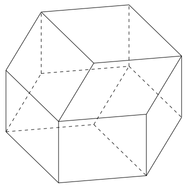
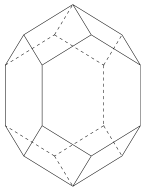


Fig. 7



*Nonprimitive parallelohedra of the* $1^{st}$ *space.* By making $\mu = 0, \mu' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 6 independent inequalities

$$\begin{array}{rcl} -\frac{1}{2}\lambda & \leq x \leq & \frac{1}{2}\lambda, \\ -\frac{1}{2}\lambda' & \leq y \leq & \frac{1}{2}\lambda', \\ -\frac{1}{2}\lambda'' & \leq z \leq & \frac{1}{2}\lambda'', \end{array}$$

which define a parallelepiped (Fig. 4).

*Nonprimirive parallelohedra of the* $2^{nd}$ *space.* By making $\mu' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 8 independent inequalities

$$\begin{array}{rcl}
-\frac{1}{2}\lambda & \leq x \leq & \frac{1}{2}\lambda, \\
-\frac{1}{2}(\lambda' + \mu) & \leq y \leq & \frac{1}{2}(\lambda' + \mu), \\
-\frac{1}{2}(\lambda'' + \mu) & \leq z \leq & \frac{1}{2}(\lambda'' + \mu), \\
-\frac{1}{2}(\lambda' + \lambda'') & \leq y + z \leq & \frac{1}{2}(\lambda' + \lambda''),
\end{array}$$

which define a prism with hexagonal base (Fig. 5).

*Nonprimirive parallelohedra of the* $3^{rd}$ *space.* By making $\lambda'' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 12 independent inequalities

$$\begin{array}{rcl}
-\frac{1}{2}(\lambda + \mu') & \leq x \leq & \frac{1}{2}(\lambda + \mu'), \\
-\frac{1}{2}(\lambda' + \mu) & \leq y \leq & \frac{1}{2}(\lambda' + \mu), \\
-\frac{1}{2}(\mu + \mu') & \leq z \leq & \frac{1}{2}(\mu + \mu'), \\
-\frac{1}{2}(\lambda' + \mu') & \leq y + z \leq & \frac{1}{2}(\lambda' + \mu'), \\
-\frac{1}{2}(\lambda + \mu) & \leq z + x \leq & \frac{1}{2}(\lambda + \mu), \\
-\frac{1}{2}(\lambda + \lambda') & \leq x + y + z \leq & \frac{1}{2}(\lambda + \lambda'),
\end{array}$$

which define a parallelogrammatic dodecahedron (Fig. 6).

*Nonprimirive parallelohedra of the* $4^{th}$ *space.* By making $\mu'' = 0$ in the inequalities (2), one will obtain 12 independent inequalities

$$\begin{array}{rcl}
-\frac{1}{2}(\lambda + \mu') & \leq x \leq & \frac{1}{2}(\lambda + \mu'), \\
-\frac{1}{2}(\lambda' + \mu) & \leq y \leq & \frac{1}{2}(\lambda' + \mu), \\
-\frac{1}{2}(\lambda'' + \mu + \mu') & \leq z \leq & \frac{1}{2}(\lambda'' + \mu + \mu'), \\
-\frac{1}{2}(\lambda' + \lambda'' + \mu') & \leq y + z \leq & \frac{1}{2}(\lambda' + \lambda'' + \mu'), \\
-\frac{1}{2}(\lambda'' + \lambda + \mu) & \leq z + x \leq & \frac{1}{2}(\lambda'' + \lambda + \mu), \\
-\frac{1}{2}(\lambda + \lambda' + \lambda'') & \leq x + y + z \leq & \frac{1}{2}(\lambda + \lambda' + \lambda''),
\end{array}$$

which define a dodecahedron in 4 hexagonal faces and in 8 parallelogrammatic face (Fig. 7). Mr. Fedorow has demonstrated that other parallelohedra in three dimensions do not exist. ¶

*Parallelohedra in four dimensions*

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The first type of primitive parallelohedra in four dimensions is characterised by the principal domain $\Delta$ of quaternary quadratic forms which is determined by the independent inequalities

$$\begin{array}{c}
\lambda_1 = a_{11} + a_{12} + a_{13} + a_{14} \geq 0, \quad \lambda_2 = a_{21} + a_{22} + a_{23} + a_{24} \geq 0, \\
\lambda_3 = a_{31} + a_{32} + a_{33} + a_{34} \geq 0, \quad \lambda_4 = a_{41} + a_{42} + a_{43} + a_{44} \geq 0, \\
\mu_1 = -a_{12} \geq 0, \quad \mu_2 = -a_{13} \geq 0, \quad \mu_3 = -a_{14} \geq 0, \\
\mu_4 = -a_{23} \geq 0, \quad \mu_5 = -a_{24} \geq 0, \quad \mu_6 = -a_{34} \geq 0.
\end{array}$$

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¶ See the Introduction

Any quaternary quadratic form

$$f(x_1, x_2, x_3, x_4)$$

belonging to the domain $\Delta$ can be determined by the equalities

$$\begin{aligned} f(x_1,x_2, x_3, x_4) = {} & \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 (x_1 - x_2)^2 + \\ & \mu_2 (x_1 - x_3)^2 + \mu_3 (x_1 - x_4)^2 + \mu_4 (x_2 - x_3)^2 + \\ & \mu_5 (x_2 - x_4)^2 + \mu_6 (x_3 - x_4)^2 . \end{aligned} \tag{1}$$

The corresponding parallelohedron is determined by 30 inequalities which one will write down in the form

$$\pm (l_1 x_1 + l_2 x_2 + l_3 x_3 + l_4 x_4) \leq \frac{1}{2} f(l_1, l_2, l_3, l_4). \tag{2}$$

The systems $\pm(l_1, l_2, l_3, l_4)$ and the corresponding values

$$f(l_1, l_2, l_3, l_4)$$

of the quadratic form (1) are given in the following table:

I$^{st}$ *type of parallelohedra*

| N | $l_1$ | $l_2$ | $l_3$ | $l_4$ | $f(l_1, l_2, l_3, l_4)$ | $-l_1$ | $-l_2$ | $-l_3$ | $-l_4$ | $2'$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | $\lambda_1 + \mu_1 + \mu_2 + \mu_3$ | $-1$ | 0 | 0 | 0 | $1'$ |
| 2 | 0 | 2 | 0 | 0 | $\lambda_2 + \mu_1 + \mu_4 + \mu_5$ | 0 | $-1$ | 0 | 0 | $2'$ |
| 3 | 0 | 0 | 1 | 0 | $\lambda_3 + \mu_2 + \mu_4 + \mu_6$ | 0 | 0 | $-1$ | 0 | $3'$ |
| 4 | 0 | 0 | 0 | 1 | $\lambda_4 + \mu_3 + \mu_5 + \mu_6$ | 0 | 0 | 0 | $-1$ | $4'$ |
| 5 | 1 | 1 | 0 | 0 | $\lambda_1 + \lambda_2 + \mu_2 + \mu_3 + \mu_4 + \mu_5$ | $-1$ | $-1$ | 0 | 0 | $5'$ |
| 6 | 1 | 0 | 1 | 0 | $\lambda_1 + \lambda_3 + \mu_1 + \mu_3 + \mu_4 + \mu_6$ | $-1$ | 0 | $-1$ | 0 | $6'$ |
| 7 | 1 | 0 | 0 | 1 | $\lambda_1 + \lambda_4 + \mu_1 + \mu_2 + \mu_5 + \mu_6$ | $-1$ | 0 | 0 | $-1$ | $7'$ |
| 8 | 0 | 1 | 1 | 0 | $\lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_5 + \mu_6$ | 0 | $-1$ | $-1$ | 0 | $8'$ |
| 9 | 0 | 1 | 0 | 1 | $\lambda_2 + \lambda_4 + \mu_1 + \mu_3 + \mu_4 + \mu_6$ | 0 | $-1$ | 0 | $-1$ | $9'$ |
| 10 | 0 | 0 | 1 | 1 | $\lambda_3 + \lambda_4 + \mu_2 + \mu_3 + \mu_4 + \mu_5$ | 0 | 0 | $-1$ | $-1$ | $10'$ |
| 11 | 1 | 1 | 1 | 0 | $\lambda_1 + \lambda_2 + \lambda_3 + \mu_3 + \mu_5 + \mu_6$ | $-1$ | $-1$ | $-1$ | 0 | $11'$ |
| 12 | 1 | 1 | 0 | 1 | $\lambda_1 + \lambda_2 + \lambda_4 + \mu_2 + \mu_4 + \mu_6$ | $-1$ | $-1$ | 0 | $-1$ | $12'$ |
| 13 | 1 | 0 | 1 | 1 | $\lambda_1 + \lambda_3 + \lambda_4 + \mu_1 + \mu_4 + \mu_5$ | $-1$ | 0 | $-1$ | $-1$ | $13'$ |
| 14 | 0 | 1 | 1 | 1 | $\lambda_1 + \lambda_3 + \lambda_4 + \mu_1 + \mu_2 + \mu_3$ | 0 | $-1$ | $-1$ | $-1$ | $14'$ |
| 15 | 1 | 1 | 1 | 1 | $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ | $-1$ | $-1$ | $-1$ | $-1$ | $15'$ |

By attributing to the parameters

$$\lambda_1,\ \lambda_2, \ldots,\ \mu_6$$

the arbitrary positive values, one will determine with the help of inequalities (2) all the primitive parallelohedra of the first type.

The primitive parallelohedra of the first type possess 120 vertices which can be divided into 12 groups composed of congruent vertices and of opposite vertices.

All these vertices are put together in the following table:

*Vertices of primitive parallelohedron of* $\mathrm{I}^{st}$ *type*

| | | | | | |
|---|---|---|---|---|---|
| I | $1\ 5\ 11\ 15$ | $1'\ 2\ 8\ 14$ | $2'\ 5'\ 3\ 10$ | $8'\ 3'\ 11'\ 4$ | $14'\ 10'\ 4'\ 15'$ |
| | $1'\ 5'\ 11'\ 15'$ | $1\ 2'\ 8'\ 4'$ | $2\ 5\ 3'\ 10'$ | $8\ 3\ 11\ 4'$ | $14\ 10\ 4\ 15$ |
| II | $1\ 5\ 3'\ 10'$ | $1'\ 2\ 6'\ 13'$ | $2'\ 5'\ 11'\ 15'$ | $6\ 11\ 3\ 4'$ | $13\ 15\ 4\ 10$ |
| | $1'\ 5'\ 3\ 10$ | $1\ 2'\ 6\ 13$ | $2\ 5\ 11\ 15$ | $6'\ 11'\ 3'\ 4$ | $13'\ 15'\ 4'\ 10'$ |
| III | $1\ 6\ 11\ 4'$ | $1'\ 3\ 8\ 7'$ | $3'\ 6'\ 2\ 13'$ | $8'\ 2'\ 11'\ 15'$ | $7\ 13\ 15\ 4$ |
| | $1'\ 6'\ 11'\ 4$ | $1\ 3'\ 8'\ 7$ | $3\ 6\ 2'\ 13$ | $8\ 2\ 11\ 15$ | $7'\ 13'\ 15'\ 4'$ |
| IV | $1\ 7\ 13\ 15$ | $1'\ 4\ 10\ 14$ | $4'\ 7'\ 3\ 8$ | $10'\ 3'\ 13'\ 2$ | $14'\ 8'\ 2'\ 15'$ |
| | $1'\ 7'\ 13'\ 15'$ | $1\ 4'\ 10'\ 14'$ | $4\ 7\ 3'\ 8'$ | $10\ 3\ 13\ 2'$ | $14\ 8\ 2\ 15$ |
| V | $1\ 5\ 12\ 15$ | $1'\ 2\ 9\ 14$ | $2'\ 5'\ 4\ 10$ | $9'\ 4'\ 12'\ 3$ | $14'\ 10'\ 3'\ 15'$ |
| | $1'\ 5'\ 12'\ 15'$ | $1\ 2'\ 9'\ 14'$ | $2\ 5\ 4'\ 10'$ | $9\ 4\ 12\ 3'$ | $14\ 10\ 3\ 15$ |
| VI | $1\ 5\ 4'\ 10'$ | $1'\ 2\ 7'\ 13'$ | $2'\ 5'\ 12'\ 15'$ | $7\ 12\ 4\ 3'$ | $13\ 15\ 3\ 10$ |
| | $1'\ 5'\ 4\ 10$ | $1\ 2'\ 7\ 13$ | $2\ 5\ 12\ 15$ | $7'\ 12'\ 4'\ 3$ | $13'\ 15'\ 3'\ 10'$ |
| VII | $1\ 7\ 12\ 3'$ | $1'\ 4\ 9\ 6'$ | $4'\ 7'\ 2\ 13'$ | $9'\ 2'\ 12'\ 15'$ | $6\ 13\ 15\ 3$ |
| | $1'\ 7'\ 12'\ 3$ | $1\ 4'\ 9'\ 6$ | $4\ 7\ 2'\ 13$ | $9\ 2\ 12\ 15$ | $6'\ 13'\ 15'\ 3'$ |
| VIII | $1\ 6\ 13\ 15$ | $1'\ 3\ 10\ 14$ | $3'\ 6'\ 4\ 9$ | $10'\ 4'\ 13'\ 2$ | $14'\ 9'\ 2'\ 15'$ |
| | $1'\ 6'\ 13'\ 15'$ | $1\ 3'\ 10'\ 14'$ | $3\ 6\ 4'\ 9'$ | $10\ 4\ 13\ 2'$ | $14\ 9\ 2\ 15$ |
| IX | $1\ 5\ 11\ 4'$ | $1'\ 2\ 8\ 7'$ | $2'\ 5'\ 3\ 12'$ | $8'\ 3'\ 11'\ 15'$ | $7\ 12\ 15\ 4$ |
| | $1'\ 5'\ 11'\ 4'$ | $1\ 2'\ 8'\ 7$ | $2\ 5\ 3'\ 12$ | $8\ 3\ 11\ 15$ | $7'\ 12'\ 15'\ 4'$ |
| X | $1\ 5\ 12\ 3'$ | $1'\ 2\ 9\ 6'$ | $2'\ 5'\ 4\ 11'$ | $9'\ 4'\ 12'\ 15'$ | $6\ 11\ 15\ 3$ |
| | $1'\ 5'\ 12'\ 3$ | $1\ 2'\ 9'\ 6$ | $2\ 5\ 4'\ 11'$ | $9\ 4\ 12\ 15$ | $6'\ 11'\ 15'\ 3'$ |
| XI | $1\ 6\ 11\ 15$ | $1'\ 3\ 8\ 14$ | $3'\ 6'\ 2\ 9$ | $8'\ 2'\ 11'\ 4$ | $14'\ 9'\ 4'\ 15'$ |
| | $1'\ 6'\ 11'\ 15'$ | $1\ 3'\ 8'\ 14'$ | $3\ 6\ 2'\ 9'$ | $8\ 2\ 11\ 4'$ | $14\ 9\ 4\ 15$ |
| XII | $1\ 7\ 12\ 15$ | $1'\ 4\ 9\ 14$ | $4'\ 7'\ 2\ 8$ | $9'\ 2'\ 12'\ 3$ | $14'\ 8'\ 3'\ 15'$ |
| | $1'\ 7'\ 12'\ 15'$ | $1\ 4'\ 9'\ 14'$ | $4\ 7\ 2'\ 8'$ | $9\ 2\ 12\ 3'$ | $14\ 8\ 3\ 15$ |

*Regulators and characteristics corresponding to the* $\mathrm{I}^{st}$ *type of parallelohedra.*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| I | $0\ 1\ 5\ 11\ 15$ | $-1\ 0\ 0\ 0$ | $\lambda_1$ | $1\ -1\ 0\ 0$ | $0\ 1\ -1\ 0$ | $0\ 0\ 1\ -1$ | $0\ 0\ 0\ 1$ |
| | | $2\ 1\ 1\ 1$ | | $(2)\ \mu_1$ | $(6)\ \mu_4$ | $(12)\ \mu_6$ | $(4')\ \lambda_4$ |
| II | $0\ 1\ 5\ 3'\ 10'$ | $-1\ 0\ 1\ 0$ | $\mu_2$ | $1\ -1\ 0\ 0$ | $0\ 1\ 0\ 0$ | $0\ 0\ -1\ 1$ | $0\ 0\ 0\ 1$ |
| | | $1\ 0\ -1\ 0$ | | $(2)\ \mu_1$ | $(14')\ \lambda_2$ | $(4')\ \mu_6$ | $(12)\ \lambda_4$ |
| III | $0\ 1\ 6\ 11\ 4'$ | $-1\ 0\ 0\ 1$ | $\mu_3$ | $1\ 0\ -1\ 0$ | $0\ -1\ 1\ 0$ | $0\ 1\ 0\ 0$ | $0\ 0\ 0\ -1$ |
| | | $1\ 0\ 0\ -1$ | | $(3)\ \mu_2$ | $(5)\ \mu_4$ | $(9')\ \lambda_2$ | $(15)\ \lambda_4$ |
| IV | $0\ 1\ 7\ 13\ 15$ | $-1\ 0\ 0\ 0$ | $\lambda_1$ | $1\ 0\ 0\ -1$ | $0\ 0\ -1\ 1$ | $0\ -1\ 1\ 0$ | $0\ 1\ 0\ 0$ |
| | | $2\ 1\ 1\ 1$ | | $(4)\ \mu_3$ | $(6)\ \mu_6$ | $(12)\ \mu_4$ | $(2')\ \lambda_2$ |
| V | $0\ 1\ 5\ 12\ 15$ | $-1\ 0\ 0\ 0$ | $\lambda_1$ | $1\ -1\ 0\ 0$ | $0\ 1\ 0\ -1$ | $0\ 0\ -1\ 1$ | $0\ 0\ 1\ 0$ |
| | | $2\ 1\ 1\ 1$ | | $(2)\ \mu_1$ | $(7)\ \mu_5$ | $(11)\ \mu_6$ | $(3')\ \lambda_3$ |
| VI | $0\ 1\ 5\ 4'\ 10'$ | $-1\ 0\ 0\ 1$ | $\mu_3$ | $1\ -1\ 0\ 0$ | $0\ 1\ 0\ 0$ | $0\ 0\ 1\ -1$ | $0\ 0\ -1\ 0$ |
| | | $1\ 0\ 0\ -1$ | | $(2)\ \mu_1$ | $(14')\ \lambda_2$ | $(3')\ \mu_6$ | $(11)\ \lambda_3$ |
| VII | $0\ 1\ 7\ 12\ 3'$ | $-1\ 0\ 1\ 0$ | $\mu_2$ | $1\ 0\ 0\ -1$ | $0\ -1\ 0\ 1$ | $0\ 1\ 0\ 0$ | $0\ 0\ -1\ 0$ |
| | | $1\ 0\ -1\ 0$ | | $(4)\ \mu_3$ | $(5)\ \mu_5$ | $(8')\ \lambda_2$ | $(15)\ \lambda_3$ |
| VIII | $0\ 1\ 6\ 13\ 15$ | $-1\ 0\ 0\ 0$ | $\lambda_1$ | $1\ 0\ -1\ 0$ | $0\ 0\ 1\ -1$ | $0\ -1\ 0\ 1$ | $0\ 1\ 0\ 0$ |
| | | $2\ 1\ 1\ 1$ | | $(3)\ \mu_2$ | $(7)\ \mu_6$ | $(11)\ \mu_5$ | $(2')\ \lambda_2$ |
| IX | $0\ 1\ 5\ 11\ 4'$ | $-1\ 0\ 0\ 1$ | $\mu_3$ | $1\ -1\ 0\ 0$ | $0\ 1\ -1\ 0$ | $0\ 0\ 1\ 0$ | $0\ 0\ 0\ -1$ |
| | | $1\ 0\ 0\ -1$ | | $(2)\ \mu_1$ | $(6)\ \mu_4$ | $(10')\ \lambda_3$ | $(15)\ \lambda_4$ |
| X | $0\ 1\ 5\ 12\ 3'$ | $-1\ 0\ 1\ 0$ | $\mu_2$ | $1\ -1\ 0\ 0$ | $0\ 1\ 0\ -1$ | $0\ 0\ 0\ 1$ | $0\ 0\ -1\ 0$ |
| | | $1\ 0\ -1\ 0$ | | $(2)\ \mu_1$ | $(7)\ \mu_5$ | $(10')\ \lambda_4$ | $(15)\ \lambda_3$ |
| XI | $0\ 1\ 6\ 11\ 15$ | $-1\ 0\ 0\ 0$ | $\lambda_1$ | $1\ 0\ -1\ 0$ | $0\ -1\ 1\ 0$ | $0\ 1\ 0\ -1$ | $0\ 0\ 0\ 1$ |
| | | $2\ 1\ 1\ 1$ | | $(3)\ \mu_2$ | $(5)\ \mu_4$ | $(13)\ \mu_5$ | $(4')\ \lambda_4$ |
| XII | $0\ 1\ 7\ 12\ 15$ | $-1\ 0\ 0\ 0$ | $\lambda_1$ | $1\ 0\ 0\ -1$ | $0\ -1\ 0\ 1$ | $0\ 1\ -1\ 0$ | $0\ 0\ 1\ 0$ |
| | | $2\ 1\ 1\ 1$ | | $(4)\ \mu_3$ | $(5)\ \mu_5$ | $(13)\ \mu_4$ | $(3')\ \lambda_3$ |

In this table, the first line of each group contains the characteristics of faces in three dimensions corresponding to the simplexes $I, II, \ldots, XII$.

The second line contains the vertices of simplexes which are contiguous to the simplexes $I, II, \ldots, XII$ by the faces, the characteristic of which are indicated above in the first line, and the regulators are indicated near by in the second line.

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Let us examine the parallelohedra in four dimensions which belong to the second type of primitive parallelohedra defined by the domain $\Delta'$ of quaternary quadratic forms. The domain $\Delta'$ is contiguous to the principal domain $\Delta$ by the face in a dimensions defined by the equation

$$\mu_1 = 0.$$

The independent regulator $\mu_1$ corresponds to the faces of simplexes

$$I,\ II,\ V,\ VI,\ IX,\ ,X.$$

All these simplexes have to be reconstructed with the help of the algorithm explained in Number 91.

One will determine the numbers $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ after the conditions

$$(2) = \vartheta_1(1) + \vartheta_2(5) + \vartheta_3(11) + \vartheta_4(15) \ \text{ and } \ \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4 = 1;$$

it becomes

$$\vartheta_0 = 1,\ \vartheta_1 = -1,\ \vartheta_2 = 1,\ \vartheta_3 = 0,\ \vartheta_4 = 0.$$

It follows that one will replace the three pairs of simplexes

$$\begin{array}{c} (0,1,5,11,15) \text{ and } (0,2,5,11,15), \\ (0,1,5,12,15) \text{ and } (0,2,5,12,15), \\ (0,1,5,11,4') \text{ and } (0,2,5,11,4') \end{array}$$

by the simplexes

$$\begin{array}{ccc} (0,1,5,11,15) & \text{and} & (0,1,2,11,15), \\ (2,1,5,12,15) & \text{and} & (0,1,2,12,15), \\ (2,1,5,11,4') & \text{and} & (0,1,2,11,4'). \end{array} \tag{3}$$

By designating the system $(1,-1,0,0)$ by the symbol $(5)$ and the system $(-1,1,0,0)$ by the symbol $(5')$, one will designate

$$\begin{array}{lll} \mathrm{I} - (0,5,1,6,13), & \mathrm{II} - (0,1,2,11,15), & \mathrm{V} - (0,5,1,7,13), \\ \mathrm{VI} - (0,1,2,12,15), & \mathrm{IX} - (0,5,1,6,9'), & \mathrm{X} - (0,1,2,11,4'). \end{array}$$

These simplexes are congruent to the new simplexes (3).

The primitive parallelohedra of the $\mathrm{II}^{nd}$ type possess 120 vertices which are brought together in the following table:

*Vertices of the primitive parallelohedron of* $\mathrm{II}^{nd}$ *type*

*Vertices of primitive parallelohedron of* $\mathrm{I}^{st}$ *type*

| | | | | | |
|---|---|---|---|---|---|
| I | $1\ 5\ 6\ 13$ | $1'\ 2'\ 3\ 10$ | $2\ 5'\ 8\ 14$ | $3'\ 8'\ 6'\ 4$ | $10'\ 14'\ 4'\ 13'$ |
| | $1'\ 5'\ 6'\ 13'$ | $1\ 2\ 3'\ 10'$ | $2'\ 5\ 8'\ 14'$ | $3\ 8\ 6\ 4'$ | $10\ 14\ 4\ 13$ |
| II | $1\ 2\ 11\ 15$ | $1'\ 5'\ 8\ 14$ | $5\ 2'\ 6\ 13$ | $8'\ 6'\ 11'\ 4$ | $14'\ 13'\ 4'\ 15'$ |
| | $1'\ 2'\ 11'\ 15'$ | $1\ 5\ 8'\ 14'$ | $5'\ 2\ 6'\ 13'$ | $8\ 6\ 11\ 4'$ | $14\ 13\ 4\ 15$ |
| III | $1\ 6\ 11\ 4'$ | $1'\ 3\ 8\ 7'$ | $3'\ 6'\ 2\ 13'$ | $8'\ 2'\ 11'\ 15'$ | $7\ 13\ 15\ 4$ |
| | $1'\ 6'\ 11'\ 4$ | $1\ 3'\ 8'\ 7$ | $3\ 6\ 2'\ 13$ | $8\ 2\ 11\ 15$ | $7'\ 13'\ 15'\ 4'$ |
| IV | $1\ 7\ 13\ 15$ | $1'\ 4\ 10\ 14$ | $4'\ 7'\ 3\ 8$ | $10'\ 3'\ 13'\ 2$ | $14'\ 8'\ 2'\ 15'$ |
| | $1'\ 7'\ 13'\ 15'$ | $1\ 4'\ 10'\ 14'$ | $4\ 7\ 3'\ 8'$ | $10\ 3\ 13\ 2'$ | $14\ 8\ 2\ 15$ |
| V | $1\ 5\ 7\ 13$ | $1'\ 2'\ 4\ 10$ | $2\ 5'\ 9\ 14$ | $4'\ 9'\ 7'\ 3$ | $10'\ 14'\ 3'\ 13'$ |
| | $1'\ 5'\ 7'\ 13'$ | $1\ 2\ 4'\ 10'$ | $2'\ 5\ 9'\ 14'$ | $4\ 9\ 7\ 3'$ | $10\ 14\ 3\ 13$ |
| VI | $1\ 2\ 12\ 15$ | $1'\ 5'\ 9\ 14$ | $5\ 2'\ 7\ 13$ | $9'\ 7'\ 12'\ 3$ | $14'\ 13'\ 3'\ 15'$ |
| | $1'\ 2'\ 12'\ 15'$ | $1\ 5\ 9'\ 14'$ | $5'\ 2\ 7'\ 13'$ | $9\ 7\ 12\ 3'$ | $14\ 13\ 3\ 15$ |
| VII | $1\ 7\ 12\ 3'$ | $1'\ 4\ 9\ 6'$ | $4'\ 7'\ 2\ 13'$ | $9'\ 2'\ 12'\ 15'$ | $6\ 13\ 15\ 3$ |
| | $1'\ 7'\ 12'\ 3$ | $1\ 4'\ 9'\ 6$ | $4\ 7\ 2'\ 13$ | $9\ 2\ 12\ 15$ | $6'\ 13'\ 15'\ 3'$ |
| VIII | $1\ 6\ 13\ 15$ | $1'\ 3\ 10\ 14$ | $3'\ 6'\ 4\ 9$ | $10'\ 4'\ 13'\ 2$ | $14'\ 9'\ 2'\ 15'$ |
| | $1'\ 6'\ 13'\ 15'$ | $1\ 3'\ 10'\ 14'$ | $3\ 6\ 4'\ 9'$ | $10\ 4\ 13\ 2'$ | $14\ 9\ 2\ 15$ |
| IX | $1\ 5\ 6\ 9'$ | $1'\ 2'\ 3\ 12'$ | $2\ 5'\ 8\ 7'$ | $3'\ 8'\ 6'\ 15'$ | $12\ 7\ 15\ 9$ |
| | $1'\ 5'\ 6'\ 9$ | $1\ 2\ 3'\ 12$ | $2'\ 5\ 8'\ 7$ | $3\ 8\ 6\ 15$ | $12'\ 7'\ 15'\ 9'$ |
| X | $1\ 2\ 11\ 4'$ | $1'\ 5'\ 8\ 7'$ | $5\ 2'\ 6\ 9'$ | $8'\ 6'\ 11'\ 15'$ | $7\ 9\ 15\ 4$ |
| | $1'\ 2'\ 11'\ 4$ | $1\ 5\ 8'\ 7$ | $5'\ 2\ 6'\ 9$ | $8\ 6\ 11\ 15$ | $7'\ 9'\ 15'\ 4'$ |
| XI | $1\ 6\ 11\ 15$ | $1'\ 3\ 8\ 14$ | $3'\ 6'\ 2\ 9$ | $8'\ 2'\ 11'\ 4$ | $14'\ 9'\ 4'\ 15'$ |
| | $1'\ 6'\ 11'\ 15'$ | $1\ 3'\ 8'\ 14'$ | $3\ 6\ 2'\ 9'$ | $8\ 2\ 11\ 4'$ | $14\ 9\ 4\ 15$ |
| XII | $1\ 7\ 12\ 15$ | $1'\ 4\ 9\ 14$ | $4'\ 7'\ 2\ 8$ | $9'\ 2'\ 12'\ 3$ | $14'\ 8'\ 3'\ 15'$ |
| | $1'\ 7'\ 12'\ 15'$ | $1\ 4'\ 9'\ 14'$ | $4\ 7\ 2'\ 8'$ | $9\ 2\ 12\ 3'$ | $14\ 8\ 3\ 15$ |

Regulators and characteristic corresponding to the II$^{nd}$-type of parallelohedra

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| I | 0 1 5 6 13 | $-1\ 0\ 0\ 0$<br>$2\ 0\ 1\ 1$ | $\lambda_1$ | $1\ 1\ -1\ 0$<br>$(2')\ \mu_1$ | $0\ -1\ 0\ 0$<br>$(15)\ \lambda_2$ | $0\ 0\ 1\ -1$<br>$(7)\ \mu_6$ | $0\ 0\ 0\ 1$<br>$(9')\ \lambda_4+\mu_1$ |
| II | 0 1 2 11 15 | $-1\ -1\ 1\ 0$<br>$1\ 1\ 0\ 0$ | $\mu_1$ | $1\ 0\ -1\ 0$<br>$(8)\ \mu_2$ | $0\ 1\ -1\ 0$<br>$(6)\ \mu_4$ | $0\ 0\ 1\ -1$<br>$(12)\ \mu_6+\mu_1$ | $0\ 0\ 0\ 1$<br>$(4')\ \lambda_4$ |
| III | 0 1 6 11 $4'$ | $-1\ 0\ 0\ 1$<br>$1\ 0\ 0\ -1$ | $\mu_3+\mu_1$ | $1\ 0\ -1\ 0$<br>$(8)\ \mu_2$ | $0\ -1\ 1\ 0$<br>$(2)\ \mu_4$ | $0\ 1\ 0\ 0$<br>$(9')\ \lambda_2+\mu_1$ | $0\ 0\ 0\ -1$<br>$(15)\ \lambda_4$ |
| IV | 0 1 7 13 15 | $-1\ 0\ 0\ 0$<br>$2\ 0\ 1\ 1$ | $\lambda_1$ | $1\ 0\ 0\ -1$<br>$(4)\ \mu_3+\mu_1$ | $0\ 0\ -1\ 1$<br>$(6)\ \mu_6$ | $0\ -1\ 1\ 0$<br>$(12)\ \mu_4+\mu_1$ | $0\ 1\ 0\ 0$<br>$(5)\ \lambda_2$ |
| V | 0 1 5 7 13 | $-1\ 0\ 0\ 0$<br>$2\ 0\ 1\ 1$ | $\lambda_1$ | $1\ 1\ 0\ -1$<br>$(2')\ \mu_1$ | $0\ -1\ 0\ 0$<br>$(15)\ \lambda_2$ | $0\ 0\ -1\ 1$<br>$(6)\ \mu_6$ | $0\ 0\ 1\ 0$<br>$(8')\ \lambda_3+\mu_1$ |
| VI | 0 1 2 12 15 | $-1\ -1\ 0\ 1$<br>$1\ 1\ 0\ 0$ | $\mu_1$ | $1\ 0\ 0\ -1$<br>$(9)\ \mu_3$ | $0\ 1\ 0\ -1$<br>$(7)\ \mu_5$ | $0\ 0\ -1\ 1$<br>$(11)\ \mu_6+\mu_1$ | $0\ 0\ 1\ 0$<br>$(3')\ \lambda_3$ |
| VII | 0 1 7 12 $3'$ | $-1\ 0\ 1\ 0$<br>$1\ 0\ -1\ 0$ | $\mu_2+\mu_1$ | $1\ 0\ 0\ -1$<br>$(9)\ \mu_3$ | $0\ -1\ 0\ 1$<br>$(2)\ \mu_5$ | $0\ 1\ 0\ 0$<br>$(8')\ \lambda_2+\mu_1$ | $0\ 0\ -1\ 0$<br>$(15)\ \lambda_3$ |
| VIII | 0 1 6 13 15 | $-1\ 0\ 0\ 0$<br>$2\ 0\ 1\ 1$ | $\lambda_1$ | $1\ 0\ -1\ 0$<br>$(3)\ \mu_2+\mu_1$ | $0\ 0\ 1\ -1$<br>$(7)\ \mu_6$ | $0\ -1\ 0\ 1$<br>$(11)\ \mu_5+\mu_1$ | $0\ 1\ 0\ 0$<br>$(5)\ \lambda_2$ |
| IX | 0 1 5 6 $9'$ | $-1\ 0\ 0\ 1$<br>$1\ -1\ 0\ -1$ | $\mu_3$ | $1\ 1\ -1\ -1$<br>$(2')\ \mu_1$ | $0\ -1\ 0\ 1$<br>$(4')\ \mu_5$ | $0\ 0\ 1\ 0$<br>$(14')\ \lambda_3$ | $0\ 0\ 0\ -1$<br>$(13)\ \lambda_4+\mu_1$ |
| X | 0 1 2 11 $4'$ | $-1\ -1\ 1\ 1$<br>$1\ 1\ 0\ 0$ | $\mu_1$ | $1\ 0\ -1\ 0$<br>$(8)\ \mu_2$ | $0\ 1\ -1\ 0$<br>$(6)\ \mu_4$ | $0\ 0\ 1\ 0$<br>$(10')\ \lambda_3+\mu_1$ | $0\ 0\ 0\ -1$<br>$(15)\ \lambda_4$ |
| XI | 0 1 6 11 15 | $-1\ 0\ 0\ 0$<br>$2\ 1\ 1\ 1$ | $\lambda_1+\mu_1$ | $1\ 0\ -1\ 0$<br>$(8)\ \mu_2$ | $0\ -1\ 1\ 0$<br>$(2)\ \mu_4$ | $0\ 1\ 0\ -1$<br>$(13)\ \mu_5+\mu_1$ | $0\ 0\ 0\ 1$<br>$(4')\ \lambda_4$ |
| XII | 0 1 7 12 15 | $-1\ 0\ 0\ 0$<br>$2\ 1\ 1\ 1$ | $\lambda_1+\mu_1$ | $1\ 0\ 0\ -1$<br>$(9)\ \mu_3$ | $0\ -1\ 0\ 1$<br>$(2)\ \mu_5$ | $0\ 1\ -1\ 0$<br>$(13)\ \mu_4+\mu_1$ | $0\ 0\ 1\ 0$<br>$(3')\ \lambda_3$ |

The domain $\Delta'$ of quaternary quadratic forms which define the second type of primitive parallelohedra in four dimensions is determined by 10 independent inequalities

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0,$$

$$\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \mu_6 \geq 0.$$

Any quaternary quadratic form belonging to the domain $\Delta'$ can be written

$$f(x_1,x_2,x_3,x_4) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 \omega + \mu_2 (x_1 - x_3)^2 +$$
$$\mu_3 (x_1 - x_4)^2 + \mu_4 (x_2 - x_3)^2 + \mu_5 (x_2 - x_4)^2 + \mu_6 (x_3 - x_4)^2$$

where

$$\omega = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 + 2x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4. \quad (4)$$

The parallelohedra belonging to the II$^{nd}$ type are determined by 30 inequalities of form (2) which are symbolically presented in the following table:

II$^{nd}$-*type of parallelohedra*

| N | $l_1$ | $l_2$ | $l_3$ | $l_4$ | $f(l_1, l_2, l_3, l_4)$ | $-l_1$ | $-l_2$ | $-l_3$ | $-l_4$ | N |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | $\lambda_1 + 2\mu_1 + \mu_2 + \mu_3$ | −1 | 0 | 0 | 0 | 1′ |
| 2 | 0 | 1 | 0 | 0 | $\lambda_2 + 2\mu 1_{+}\mu_4 + \mu_5$ | 0 | −1 | 0 | 0 | 2′ |
| 3 | 0 | 0 | 1 | 0 | $\lambda_3 + 2\mu_1 + \mu_2 + \mu_4 + \mu_6$ | 0 | 0 | −1 | 0 | 3′ |
| 4 | 0 | 0 | 0 | 1 | $\lambda_4 + 2\mu_1 + \mu_3 + \mu_5 + \mu_6$ | 0 | 0 | 0 | −1 | 4′ |
| 5 | 1 | −1 | 0 | 0 | $\lambda_1 + \lambda_2 + 2\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$ | −1 | 1 | 0 | 0 | 5′ |
| 6 | 1 | 0 | 1 | 0 | $\lambda_1 + \lambda_3 + 2\mu_1 + \mu_3 + \mu_4 + \mu_6$ | −1 | 0 | −1 | 0 | 6′ |
| 7 | 1 | 0 | 0 | 1 | $\lambda_1 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_5 + \mu_6$ | −1 | 0 | 0 | −1 | 7′ |
| 8 | 0 | 1 | 1 | 0 | $\lambda_2 + \lambda_3 + 2\mu_1 + \mu_2 + \mu_5 + \mu_6$ | 0 | −1 | −1 | 0 | 8′ |
| 9 | 0 | 1 | 0 | 1 | $\lambda_2 + \lambda_4 + 2\mu_1 + \mu_3 + \mu_4 + \mu_6$ | 0 | −1 | 0 | −1 | 9′ |
| 10 | 0 | 0 | 1 | 1 | $\lambda_3 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$ | 0 | 0 | −1 | −1 | 10′ |
| 11 | 1 | 1 | 1 | 0 | $\lambda_1 + \lambda_2 + \lambda_3 + 4\mu_1 + \mu_3 + \mu_5 + \mu_6$ | −1 | −1 | −1 | 0 | 11′ |
| 12 | 1 | 1 | 0 | 1 | $\lambda_1 + \lambda_2 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_4 + \mu_6$ | −1 | −1 | 0 | −1 | 12′ |
| 13 | 1 | 0 | 1 | 1 | $\lambda_1 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_4 + \mu_5$ | −1 | 0 | −1 | −1 | 13′ |
| 14 | 0 | 1 | 1 | 1 | $\lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_3$ | 0 | −1 | −1 | −1 | 14′ |
| 15 | 1 | 1 | 1 | 1 | $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1$ | −1 | −1 | −1 | −1 | 15′ |

116 Let us find the domain $\Delta''$ of quaternary quadratic forms which is contiguous to the domain $\Delta'$ by the face in 9 dimensions defined by the equation

$$\mu_6 = 0.$$

The independent regulator $\mu_6$ corresponds to the faces of simplexes

$$I,\ IV,\ V,\ VIII.$$

All these simplexes have to be reconstructed after the algorithm explained in Number 91. One will determine, to this effect, the number

$$\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$$

after the condition

$$\vartheta_1(1) + \vartheta_2(5) + \vartheta_3(6) + \vartheta_4(13) \quad \text{and} \quad \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4; \quad (5)$$

it becomes

$$\vartheta_0 = 0, \quad \vartheta_1 = 1, \quad \vartheta_2 = 0, \quad \vartheta_3 = -1, \quad \vartheta_4 = 1.$$

One concludes that the two pairs of simplexes

$$(0, 1, 5, 6, 13) \text{ and } (0, 1, 5, 7, 13) \text{ and}$$
$$(0, 1, 7, 13, 15) \text{ and } (0, 1, 6, 13, 15)$$

have to be replaced by the new simplexes

$$\text{I} - (0, 7, 5, 6, 13) \quad \text{and} \quad \text{V} - (0, 1, 5, 6, 7)$$
$$\text{IV} -(0, 6, 7, 13, 15) \quad \text{and} \quad \text{VIII} - (0, 1, 7, 6, 15).$$

By designating the system $(0,0,1,-1)$ by the symbol $(10)$ and the system $(0,0,-1,1)$ by the symbol $(10')$ one will determine all the vertices of primitive parallelohedra belonging to the new type as follows.

Vertices of primitive parallelohedron of III$^{rd}$-type.

| | | | | | |
|---|---|---|---|---|---|
| I | $7\ 5\ 6\ 13$ | $7'\ 9'\ 10\ 3$ | $9\ 5'\ 8\ 14$ | $10'\ 8'\ 6'\ 4$ | $3'\ 14'\ 4'\ 13'$ |
| | $7'\ 5'\ 6'\ 13'$ | $7\ 9\ 10'\ 3'$ | $9'\ 5\ 8'\ 14'$ | $10\ 8\ 6\ 4'$ | $3\ 14\ 4\ 13$ |
| II | $1\ 2\ 11\ 15$ | $1'\ 5'\ 8\ 14$ | $5\ 2'\ 6\ 13$ | $8'\ 6'\ 11'\ 4$ | $14'\ 13'\ 4'\ 15'$ |
| | $1'\ 2'\ 11'\ 15'$ | $1\ 5\ 8'\ 14'$ | $5'\ 2\ 6'\ 13'$ | $8\ 6\ 11\ 4'$ | $14\ 13\ 4\ 15$ |
| III | $1\ 6\ 11\ 4'$ | $1'\ 3\ 8\ 7'$ | $3'\ 6'\ 2\ 13'$ | $8'\ 2'\ 11'\ 15'$ | $7\ 13\ 15\ 4$ |
| | $1'\ 6'\ 11'\ 4$ | $1\ 3'\ 8'\ 7$ | $3\ 6\ 2'\ 13$ | $8\ 2\ 11\ 15$ | $7'\ 13'\ 15'\ 4'$ |
| IV | $6\ 7\ 13\ 15$ | $6'\ 10'\ 4\ 9$ | $10\ 7'\ 3\ 8$ | $4'\ 3'\ 13'\ 2$ | $9'\ 8'\ 2'\ 15'$ |
| | $6'\ 7'\ 13'\ 15'$ | $6\ 10\ 4'\ 9'$ | $10'\ 7\ 3'\ 8'$ | $4\ 3\ 13\ 2'$ | $9\ 8\ 2\ 15$ |
| V | $1\ 5\ 6\ 7$ | $1'\ 2'\ 3\ 4$ | $2\ 5'\ 8\ 9$ | $3'\ 8'\ 6'\ 10'$ | $4'\ 9'\ 10\ 7'$ |
| | $1'\ 5'\ 6'\ 7'$ | $1\ 2\ 3'\ 4'$ | $2'\ 5\ 8'\ 9'$ | $3\ 8\ 6\ 10$ | $4\ 9\ 10'\ 7$ |
| VI | $1\ 2\ 12\ 15$ | $1'\ 5'\ 9\ 14$ | $5\ 2'\ 7\ 13$ | $9'\ 7'\ 12'\ 3$ | $14'\ 13'\ 3'\ 15'$ |
| | $1'\ 2'\ 12'\ 15'$ | $1\ 5\ 9'\ 14'$ | $5'\ 2\ 7'\ 13'$ | $9\ 7\ 12\ 3'$ | $14\ 13\ 3\ 15$ |
| VII | $1\ 7\ 12\ 3'$ | $1'\ 4\ 9\ 6'$ | $1\ 4'\ 9'\ 6$ | $4'\ 7'\ 2\ 13'$ | $9'\ 2'\ 12'\ 15'$ |
| | $6\ 13\ 15\ 3$ | $1'\ 7'\ 12'\ 3$ | $4\ 7\ 2\ 13$ | $9\ 2\ 12\ 15$ | $6'\ 13'\ 15'\ 3'$ |
| VIII | $1\ 7\ 6\ 15$ | $1'\ 4\ 3\ 14$ | $4'\ 7'\ 10\ 8$ | $3'\ 10'\ 6'\ 9$ | $14'\ 8'\ 9'\ 15'$ |
| | $1'\ 7'\ 6'\ 15'$ | $1\ 4'\ 3'\ 14'$ | $4\ 7\ 10'\ 8'$ | $3\ 10\ 6\ 9'$ | $14\ 8\ 9\ 15$ |
| IX | $1\ 5\ 6\ 9'$ | $1'\ 2'\ 3\ 12'$ | $2\ 5'\ 8\ 7'$ | $3'\ 8'\ 6'\ 15'$ | $12\ 7\ 15\ 9$ |
| | $1'\ 5'\ 6'\ 9$ | $1\ 2\ 3'\ 12$ | $2'\ 5\ 8'\ 7$ | $3\ 8\ 6\ 15$ | $12'\ 7'\ 15'\ 9'$ |
| X | $1\ 2\ 11\ 4'$ | $1'\ 5'\ 8\ 7'$ | $5\ 2'\ 6\ 9'$ | $8'\ 6'\ 11'\ 15'$ | $7\ 9\ 15\ 4$ |
| | $1'\ 2'\ 11'\ 4$ | $1\ 5\ 8'\ 7$ | $5'\ 2\ 6'\ 9$ | $8\ 6\ 11\ 15$ | $7'\ 9'\ 15'\ 4'$ |
| XI | $1\ 6\ 11\ 15$ | $1'\ 3\ 8\ 14$ | $3'\ 6'\ 2\ 9$ | $8'\ 2'\ 11'\ 4$ | $14'\ 9'\ 4'\ 15'$ |
| | $1'\ 6'\ 11'\ 15'$ | $1\ 3'\ 8'\ 14'$ | $3\ 6\ 2'\ 9'$ | $8\ 2\ 11\ 4'$ | $14\ 9\ 4\ 14$ |
| XII | $1\ 7\ 12\ 15$ | $1'\ 4\ 9\ 14$ | $4'\ 7'\ 2\ 8$ | $9'\ 2'\ 12'\ 3$ | $14'\ 8'\ 3'\ 15'$ |
| | $1'\ 7'\ 12'\ 15'$ | $1\ 4'\ 9'\ 14'$ | $4\ 7\ 2'\ 8'$ | $9\ 2\ 12\ 3'$ | $14\ 8\ 3\ 15$ |

Regulators and characteristics corresponding to the III$^{rd}$-type parallelohedra

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| I | 0 7 5 6 13 | –1 0 0 0<br>2 0 1 1 | $\lambda_1$ | 1 1 –1 0<br>(2′) $\mu_1$ | 0 –1 0 0<br>(15) $\lambda_2$ | 1 1 0 –1<br>(2′) $\mu_1$ | –1 –1 1 1<br>(1) $\mu_6$ |
| II | 0 1 2 11 15 | –1 –1 1 0<br>1 1 0 1 | $\mu_1$ | 1 0 –1 0<br>(8) $\mu_2$ | 0 1 –1 0<br>(6) $\mu_4$ | 0 0 1 –1<br>(12) $\mu_1$ | 0 0 0 1<br>(4′) $\lambda_4$ |
| III | 0 1 6 11 4′ | –1 0 0 1<br>1 0 1 –1 | $\mu_3+\mu_1$ | 1 0 –1 0<br>(8) $\mu_2$ | 0 –1 1 0<br>(2) $\mu_4$ | 0 1 0 0<br>(9′) $\lambda_2+\mu_1$ | 0 0 0 –1<br>(15) $\lambda_4$ |
| IV | 0 6 7 13 15 | –1 0 0 0<br>2 0 1 1 | $\lambda_1$ | 1 0 0 –1<br>(4) $\mu_3+\mu_1$ | 1 0 –1 0<br>(3) $\mu_2+\mu_1$ | –1 –1 1 1<br>(1) $\mu_6$ | 0 1 0 0<br>(5) $\lambda_2$ |
| V | 0 1 5 6 7 | –1 0 0 0<br>2 0 1 1 | $\lambda_1$ | 1 1 –1 –1<br>(13) $\mu_6$ | 0 –1 0 0<br>(15) $\lambda_2$ | 0 0 1 0<br>(8′) $\lambda_3+\mu_1$ | 0 0 0 1<br>(9′) $\lambda_4+\mu_1$ |
| VI | 0 1 2 12 15 | –1 –1 0 1<br>1 1 1 0 | $\mu_1$ | 1 0 0 –1<br>(9) $\mu_3$ | 1 1 0 –1<br>(7) $\mu_5$ | 0 0 –1 1<br>(11) $\mu_1$ | 0 0 1 0<br>(3′) $\lambda_3$ |
| VII | 0 1 7 12 3′ | –1 0 0 0<br>1 0 –1 1 | $\mu_2+\mu_1$ | 1 0 0 –1<br>(9) $\mu_3$ | 0 –1 0 1<br>(2) $\mu_5$ | 0 1 0 0<br>(8′) $\lambda_2+\mu_1$ | 0 0 –1 0<br>(15) $\lambda_3$ |
| VIII | 0 1 7 6 15 | –1 0 0 0<br>2 0 1 1 | $\lambda_1$ | 1 1 –1 –1<br>(13) $\mu_6$ | 0 –1 0 1<br>(11) $\mu_5+\mu_1$ | 0 –1 1 0<br>(12) $\mu_4+\mu_1$ | 0 1 0 0<br>(5) $\lambda_2$ |
| IX | 0 1 5 6 9′ | –1 0 0 1<br>1 –1 0 –1 | $\mu_3$ | 1 1 –1 –1<br>(2′) $\mu_1+\mu_6$ | 0 –1 0 1<br>(4′) $\mu_5$ | 0 0 1 0<br>(14′) $\lambda_3$ | 0 0 0 –1<br>(7) $\lambda_4+\mu_1$ |
| X | 0 1 2 11 4′ | –1 –1 1 1<br>1 1 0 0 | $\mu_1+\mu_6$ | 1 0 –1 0<br>(8) $\mu_2$ | 0 1 –1 0<br>(6) $\mu_4$ | 0 0 1 0<br>(3′) $\lambda_3+\mu_1$ | 0 0 0 –1<br>(15) $\lambda_4$ |
| XI | 0 1 6 11 15 | –1 0 0 0<br>2 1 1 1 | $\lambda_1+\mu_1$ | 1 0 –1 0<br>(8) $\mu_2$ | 0 –1 1 0<br>(2) $\mu_4$ | 0 1 0 –1<br>(7) $\mu_5+\mu_1$ | 0 0 0 1<br>(4′) $\lambda_4$ |
| XII | 0 1 7 12 15 | –1 0 0 0<br>2 1 1 1 | $\lambda_1+\mu_1$ | 1 0 0 –1<br>(9) $\mu_3$ | 0 –1 0 1<br>(2) $\mu_5$ | 0 1 –1 0<br>(6) $\mu_4+\mu_1$ | 0 0 1 0<br>(3′) $\lambda_3$ |

The independent regulators are expressed by the formulae

$$\lambda_1 = a_{11}+a_{13}+a_{14}+a_{34}, \lambda_2 = a_{22}+a_{23}+a_{24}+a_{34}, \lambda_3 = a_{31}+a_{32}+a_{33}+a_{34},$$

$$\lambda_4 = a_{41} + a_{42} + a_{43} + a_{44}, \mu_1 = a_{12} - a_{34}, \mu_2 = -a_{13} - a_{12},$$

$$\mu_3 = -a_{14} - a_{12}, \mu_4 = -a_{23} - a_{12}, \mu_5 = -a_{24} - a_{12}, \mu_6 = a_{34}.$$

The domain $\Delta''$ of quaternary quadratic forms which defines the third type of primitive parallelohedra in four dimensions is determined by 10 independent inequalities

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0,$$

$$\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \mu_6 \geq 0.$$

Each quaternary quadratic form belonging to the domain $\Delta''$ can be written

$$f(x_1, x_2, x_3, x_4) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 \omega + \mu_2 (x_1 - x_3)^2$$
$$+ \mu_3 (x_1 - x_4)^2 + \mu_4 (x_2 - x_3)^2 + \mu_5 (x_2 - x_4)^2 + \mu_6 (x_1 + x_2 - x_3 - x_4)^2,$$

the form $\omega$ being defined by the equality (4). The parallelohedra belonging to the III$^{rd}$ type are determined by 30 inequalities of the form (2) which are symbolically presented in the following table:

III$^{rd}$ type of parallelohedra

| $N$ | $l_1 \; l_2 l_3 \, l_4$ | $f(l_1, l_2, l_3, l_4)$ | $-l_1 - l_2 - l_3 - l_4$ | $N$ |
|---|---|---|---|---|
| 1 | 1 0 0 0 | $\lambda_1 + 2\mu_1 + \mu_2 + \mu_3 + \mu_6$ | −1 0 0 0 | 1′ |
| 2 | 0 1 0 0 | $\lambda_2 + 2\mu_1 + \mu_4 + \mu_5 + \mu_6$ | 0 −1 0 0 | 2′ |
| 3 | 0 0 1 0 | $\lambda_3 + 2\mu_1 + \mu_2 + \mu_4 + \mu_6$ | 0 0 −1 0 | 3′ |
| 4 | 0 0 0 1 | $\lambda_4 + 2\mu_1 + \mu_3 + \mu_5 + \mu_6$ | 0 0 0 −1 | 4′ |
| 5 | 1 −1 0 0 | $\lambda_1 + \lambda_2 + 2\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$ | −1 1 0 0 | 5′ |
| 6 | 1 0 1 0 | $\lambda_1 + \lambda_3 + 2\mu_1 + \mu_3 + \mu_4$ | −1 0 −1 0 | 6′ |
| 7 | 1 0 0 1 | $\lambda_1 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_5$ | −1 0 0 −1 | 7′ |
| 8 | 0 1 1 0 | $\lambda_2 + \lambda_3 + 2\mu_1 + \mu_2 + \mu_5$ | 0 −1 −1 0 | 8′ |
| 9 | 0 1 0 1 | $\lambda_2 + \lambda_4 + 2\mu_1 + \mu_3 + \mu_4$ | 0 −1 0 −1 | 9′ |
| 10 | 0 0 1 −1 | $\lambda_3 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$ | 0 0 −1 1 | 10′ |
| 11 | 1 1 1 0 | $\lambda_1 + \lambda_2 + \lambda_3 + 4\mu_1 + \mu_3 + \mu_5 + \mu_6$ | −1 −1 −1 0 | 11′ |
| 12 | 1 1 0 1 | $\lambda_1 + \lambda_2 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_4 + \mu_6$ | −1 −1 0 −1 | 12′ |
| 13 | 1 0 1 1 | $\lambda_1 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_4 + \mu_5 + \mu_6$ | −1 0 −1 −1 | 13′ |
| 14 | 0 1 1 1 | $\lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_3 + \mu_6$ | 0 −1 −1 −1 | 14′ |
| 15 | 1 1 1 1 | $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1$ | −1 −1 −1 −1 | 15′ |

117 We have determined three domains $\Delta, \Delta', \Delta''$ which characterise three types of primitive parallelohedra in four dimensions.

*Theorem. The set* $(\Delta)$ *of domains of quaternary quadratic forms is composed of three different classes which can be represented by the domain* $\Delta, \Delta', \Delta''$.

In my first mémoire cited, it has been demonstrated that the set $(R)$ of domains of quaternary quadratic forms corresponding to the perfect quaternary quadratic forms is composed of two classes represented by the principal domain $R$ and by a domain $R$, determined by the equalities

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = {} & \rho_1 x_1^2 + \rho_2 x_2^2 + \rho_3 x_3^2 + \rho_4 x_4^2 + \rho_5 (x_1 - x_3)^2 + \\ & \rho_6 (x_1 - x_4)^2 + \rho_7 (x_1 - x_3)^2 + \rho_8 (x_2 - x_4)^2 + \rho_9 (x_3 - x_4)^2 + \\ & \rho_{10} (x_1 + x_2 - x_3)^2 + \rho_{11} (x_1 + x_2 - x_4)^2 + \rho_{12} (x_1 + x_2 - x_3 - x_4)^2 \end{aligned}$$

where $\rho_1, \rho_2, \ldots, \rho_{12}$ are positive arbitrary parameters or zeros.

The domain $R$, corresponds to a perfect form

$$\varphi_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4. \quad (5)$$

In the mémoire cited, it has been demonstrated that all the faces in 9 dimensions of domain $R$, are equivalent to two faces characterised: one by the quadratic form

$$x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, (x_3 - x_4)^2$$

and the other by the quadratic form

$$\begin{gathered} x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, \\ (x_1 + x_2 - x_3 - x_4)^2. \end{gathered}$$

The first face verifies the equation

$$a_{12} = 0$$

and the second face verifies the equation

$$a_{12} - a_{34} = 0$$

The form $\omega$ determined by the formula (4) characterise the axis of the domain $(R)$ which does not change when one transforms the domain $R$ into itself.

One concludes that the domain $R$ can be partitioned into groups all of which are equivalent to the two domains $\Delta'$ and $\Delta''$ obtained. This results in that the principal domain $\Delta$ and the two domains $\Delta'$ and $\Delta''$ can not be equivalent.

The theorem introduced is thus demonstrated.

By not considering as different the equivalent types of parallelohedra one can say that there are only three different types of primitive parallelohedra in the space in four dimensions.

By calling reduced the positive quadratic forms which belong to the domains $\Delta$, $\Delta'$ and $\Delta''$ one obtains a new method of reduction of quaternary positive quadratic forms which presents a modification of the method due to Mr. Charve. ‡

In effect, following the method of Mr. Charve, one calls reduced the quaternary positive quadratic forms belonging to one of three simple domains $R, R'$ and $R''$. The first two domain $R$ and $R'$ coincide with the domains $\Delta$ and $\Delta'$ and it is only the third domain $R''$ of Mr. Charve which differs from the domain $\Delta''$. Any form belonging to the domain $\Delta''$ is equivalent to a form belonging to the domain $R''$ and vice-versa.

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By examining the two tables which contain the characteristics of faces of simplexes which define the $2^{nd}$ and the $3^{rd}$ type of primitive parallelohedra, one will observe that these characteristic coincide for the two types and are represented by the linear forms

$$\begin{aligned}
&\pm x_1, \pm x_2, \pm x_3, \pm x_4, \pm(x_1 - x_3),\\
&\pm (x_1 - x_4), \pm(x_2 - x_3), \pm(x_2 - x_4), \pm(x_3 - x_4),\\
&\pm (x_1 + x_2 - x_3), \pm(x_1 + x_2 - x_4), \pm(x_1 + x_2 - x_3 - x_4).
\end{aligned}$$

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‡ See the introduction

It is remarkable that these linear forms define the set of representations of the minimum of the perfect form $\varphi_1$ determined by the equality (5).

By virtue of that which has previously been mentioned, one can affirm that the coincidence noticed appears as the characteristics of faces of all the primitive parallelohedra in $2, 3$ and $4$ dimensions.

It would be interesting to find out whether this is only a coincidence or whether there really exists a relation between the two problems which seem to be different: between the problem of the uniform partition of the space with the help of congruent convex polyhedra and the study of perfect positive quadratic forms.

End of the second mémoire

[in German]

Immediately after the first sheet of this significant work was set, we received the grievous tidings that your author of the science has been taken away by Death. The editor had in the best power seen to it that this last work of he who so early departed for the other side was checked over with utmost care.

Marburg, 19th June 1909

K. Hensel

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